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Abstract

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A LOCAL CRITERION FOR THE EXISTENCE OF WAVE OPERATORS

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Conditions for the existence of wave operators have been considered in a number of works ([1], [13]) devoted to the abstract theory of scattering.* Here we propose a new general criterion for the existence of wave operators, which contains all those obtained previously, as well as some new useful simple criteria. In contrast to the usual definitions, here we use a "local" definition of wave operators and of the scattering operator, associating them with some subset of points of the spectrum. It is also convenient for us to distinguish strong and weak wave operators. The main result of the paper is Theorem 4. At the same time, it is first clarified that each of the three requirements in the hypotheses of this theorem is connected with a certain property of the wave operator. Such a separation of properties is apparently of interest in itself. We also note that the method of the present paper is a further development of the method of paper ([13]), where for the first time a direct "stationary" proof was given of the simplest criterion for the existence of wave operators, due to M. Rosenblum ([1]) and T. Kato ([3]). In this way it is possible to avoid the reduction of the problem to the Rosenblum–Kato case, which is usually used, and to obtain more general results.

1. Everywhere in what follows H_0, H are self-adjoint operators in a Hilbert space $\mathfrak{H}$; R_z^0, R_z and $E_0(G), E(G)$ are their corresponding resolvents and spectral measures; P_0, P are the projectors onto the absolutely continuous subspaces $\mathfrak{G}_0, \mathfrak{G}$ of the operators H_0, H, respectively. The domain of definition of an operator K will be denoted by $\mathfrak{D}(K)$, and the adjoint operator by K^*. The class of all nuclear operators is denoted by $\mathfrak{S}_1$, and of all completely continuous ones by $\mathfrak{S}_\infty$.

Let G be a Borel set on the real axis. The strong limit

$$W_\pm(H, H_0; G) = \lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)E_0(G)P_0, \tag{1}$$

if it exists, is called the wave operator for the ordered pair of operators H_0, H on the set G. In the case $G = (-\infty, +\infty)$ this definition coincides with the usual definition of the wave operators $W_\pm(H, H_0)$. Each of the operators $W_\pm(H, H_0; G)$

maps $E_0(G)\mathfrak{G}_0$ isometrically into $E(G)\mathfrak{G}$, and moreover

$$W_\pm(H, H_0; G)H_0 = HW_\pm(H, H_0; G). \tag{2}$$

The wave operator $W_\pm(H, H_0; G)$ is called complete if the range of its values coincides with $E(G)\mathfrak{G}$.

Along with (1), we define the weak wave operators by means of the weak limit

$$\dot{W}_\pm(H, H_0; G) = w - \lim_{t\to\pm\infty} E(G)P \exp(iHt) \exp(-iH_0 t)E_0(G)P_0 \tag{3}$$

and note that the operators $\dot{W}_\pm$ also satisfy an intertwining relation of the form (2). They are, however, generally speaking, not isometric–

* For the basic definitions see, for example, ([9]) or ([5]).

ical. From the existence of $W_\pm(H, H_0; G)$, obviously, there follows the existence of $\mathring{W}_\pm(H, H_0; G) = W_\pm(H, H_0; G)$. The following simple lemma is also valid.

Lemma. In order that the wave operator (1) exist, it is necessary and sufficient that the weak wave operator (3) exist and that the latter be isometric on $E_0(G)\mathfrak{H}_0$.

Let us also note that, simultaneously with (3), there exist the weak wave operators

$$\mathring{W}_\pm(H_0, H; G) = \mathring{W}^*_\pm(H, H_0; G).$$

2. Below the Borel set G is assumed to be bounded.

Theorem 1. The weak wave operators $\mathring{W}_\pm(H, H_0; G)$ exist if the condition

$$HE(G)E_0(G) - E(G)H_0E_0(G) \in \mathfrak{S}_1 \tag{α}$$

is satisfied.

Condition (α) ensures the existence of weak wave operators not only for the operators H_0, H, but also for some of their functions. More precisely, the following is true.

Theorem 2. Let condition (α) be fulfilled for the finite segment $G = [a, b]$. Let $\varphi(\lambda)$ be a nondecreasing function which everywhere on G has a positive derivative satisfying the Lipschitz condition*, and let $\widetilde{H}_0 = \varphi(H_0)$, $\widetilde{H} = \varphi(H)$. Then condition (α) is also fulfilled for the pair $\widetilde{H}_0, \widetilde{H}$ on the segment $\widetilde{G} = [\varphi(a), \varphi(b)]$, and, consequently, there exist the wave operators $\mathring{W}_\pm(\widetilde{H}, \widetilde{H}_0; \widetilde{G})$. Moreover,

$$\mathring{W}_\pm(\widetilde{H}, \widetilde{H}_0; \widetilde{G}) = \mathring{W}_\pm(H, H_0; G). \tag{4}$$

3. Relation (4), together with the lemma, makes it possible to establish the existence of strong wave operators for the pair H_0, H, if it has already been established for the pair $\widetilde{H}_0, \widetilde{H}$. Theorem 3 below makes it possible to establish the existence of strong wave operators directly.

Denote by G' the complement of G. We shall say that, for the pair of operators H_0, H, condition (β) is satisfied on G if one can specify a sequence of Borel sets $G_n \subset G$ $(n = 0, 1, 2, ...)$ such that

$$G = G_0 + \bigcup_{n=1}^{\infty} G_n, \qquad G_0 \text{ has Lebesgue measure zero and}$$

$$E(G')E_0(G_n) \in \mathfrak{S}_\infty \quad (n = 1, 2, ...). \tag{β}$$

Theorem 3. If, for the pair of operators H_0, H on the set G, conditions (α) and (β) are fulfilled, then there exist the strong wave operators $W_\pm(H, H_0; G)$.

Remark. Verification of condition (β) may be facilitated if there exists a nondecreasing function $\psi(\lambda)$ such that $\psi(-\infty) = -\infty$, $\psi(+\infty) = +\infty$, and $\mathfrak{D}(\psi(H_0)) = \mathfrak{D}(\psi(H))$. Then instead of (β) it suffices to verify the condition

$$E(G' \cap [-N, N])E_0(G_n) \in \mathfrak{S}_\infty \quad (n = 1, 2, ...)$$

for arbitrarily large values of N.

Let us note that, in contrast to condition (α), in condition (β) the operators H_0 and H enter asymmetrically. A more complete result is obtained if we assume condition (β) to be fulfilled also for the pair H, H_0.

Theorem 4. Let conditions (α) and (β) be fulfilled on the set G for each of the pairs of operators H_0, H and H, H_0. Then there exist the complete strong wave operators $W_\pm(H, H_0; G)$ and $W_\pm(H_0, H; G)$.

Theorem 4 in principle makes it possible to establish the existence of the operators $W_\pm(H, H_0; G)$ also in cases where the wave operators $W_\pm(H, H_0)$ do not exis–

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* The smoothness requirement on the function $\varphi(\lambda)$ on G could be somewhat weakened, but we shall not dwell on this here. Outside G the function $\varphi(\lambda)$ may be extended arbitrarily, but with monotonicity preserved.

exist. At the same time, the existence of the latter follows from Theorem 4 if the real axis can be covered, up to a set of measure zero, by bounded Borel sets for which the conditions of the theorem are satisfied.

4. If the complete wave operators (1) exist, then the operator

$$S(H, H_0; G) = W_+^*(H, H_0; G)\, W_-(H, H_0; G) \tag{5}$$

is unitary in the subspace $E_0(G)\mathfrak{H}_0$ and commutes there with H_0. We shall call the operator (5) the S-operator (scattering operator) for the pair of operators H_0, H and the set G. The usual definition of the S-operator corresponds to the case $G = (-\infty, +\infty)$. Decompose the subspace $E_0(G)\mathfrak{H}_0$ into a continuous direct sum of Hilbert spaces $\mathfrak{h}_\lambda$ so that the part of the operator H_0 in $E_0(G)\mathfrak{H}_0$ becomes the operator of multiplication by λ. To the operator $S(H, H_0; G)$ in this decomposition there corresponds a measurable family $S_\lambda(H, H_0)$ of unitary operators in $\mathfrak{h}_\lambda$. This family is defined for almost all $\lambda \in G \cap \Lambda$, where Λ is the absolutely continuous spectrum of the operator H_0. Generalizing the corresponding definition from ([7], [10]), we shall call the operator $S_\lambda(H, H_0)$ the scattering suboperator (S-matrix) for the pair H, H_0. It is easy to see that the S-matrix in fact does not depend on the domain G. Denote by I_λ the identity operator in $\mathfrak{h}_\lambda$. The following assertion holds, first established in ([7]) (see also ([10], [13])) under more restrictive assumptions.

Theorem 5. *Under the conditions of Theorem 4, for almost all $\lambda \in G \cap \Lambda$*

$$S_\lambda(H, H_0) - I_\lambda \in \mathfrak{S}_1.$$

5. We now dwell on some consequences of Theorem 4 which give practically convenient criteria for the existence of the complete wave operators $W_\pm(H, H_0)$. We first note the following result, obtained earlier by the author ([8], [9]).

Let $\varphi(\lambda)$ be an admissible* function for the pair of operators H_0, H, let $\widetilde{H}_0 = \varphi(H_0)$, $\widetilde{H} = \varphi(H)$, and let the condition

$$\widetilde{H} - \widetilde{H}_0 \in \mathfrak{S}_1 \tag{6}$$

be satisfied, or, more generally,

$$(\widetilde{H} - iE)^{-1} - (\widetilde{H}_0 - iE)^{-1} \in \mathfrak{S}_1. \tag{7}$$

Then the complete wave operators $W_\pm(H, H_0)$ exist, and moreover

$$W_\pm(H, H_0) = W_\pm(\widetilde{H}, \widetilde{H}_0). \tag{8}$$

The proof of this fact, given in ([9]), is based on the construction of the weak wave operators $\widetilde{W}_\pm(H, H_0)$, $\widetilde{W}_\pm(\widetilde{H}, \widetilde{H}_0)$ and on the proof of their coincidence. Thus (see the lemma in Sec. 1) the matter was reduced to the Rosenblum–Kato theorem ([1], [3]) (under condition (6)) or to the more general theorem of M. G. Krein and the author ([7]) (under condition (7)). In a recent paper ([11]) T. Kato gave another ("nonstationary") proof of the result under discussion, but it was also based on relation (8)**, which reduces the matter to the simplest case. In addition, T. Kato observed that one may dispense with the univalence of the admissible function. Instead, one should assume the existence of a sequence of admissible functions $\varphi_n(\lambda)$, each of which is univalent only on the interval $(-n, n)$, and such that

$$\varphi_n(H) - \varphi_n(H_0) \in \mathfrak{S}_1.$$

* For the precise definition of an admissible function, see ([9]) or ([10]). Here we note only that an admissible function is univalent, i.e., the mapping it defines is one-to-one.

** T. Kato proposed calling relation (8) the invariance principle. In this connection we note that the broadest formulation of the invariance principle is given by Theorem 2.

Let us now note that the results presented here (and even some more general ones) can be derived directly from Theorem 4, without relying on the invariance principle and the Rosenblum–Kato theorem. In this case the proof becomes in many respects more transparent.

6. Theorem 4 also makes it possible to obtain certain new useful criteria that do not follow from the results of §5. In particular, the following is valid.

Theorem 6. *Let $\mathfrak{D}(H) = \mathfrak{D}(H_0)$ and*

$$R_z^k (H - H_0)(R_z^0)^l \in \mathfrak{S}_1 \quad (k \geq 0,\ l \geq 0,\ k + l > 0). \tag{9}$$

Then the complete wave operators $W_\pm(H, H_0)$ exist.

For $k = l$, Theorem 6 contains the criteria of the works ([12], [13]). More interesting, however, is the "nonsymmetric" special case $k = 0$, $l > 0$. In particular, it makes it possible to study in the space $L_2(\mathcal{E}_m^\xi)$ the operator $H = -\Delta + q(x)$ $(H_0 = -\Delta)$ without increasing the smoothness of the decreasing potential $q(x)$, as the dimension m grows.

If $\mathfrak{D}(H) \neq \mathfrak{D}(H_0)$, then condition (9) may be replaced by the conditions

$$(R_z - R_z^0)(R_z^0)^l \in \mathfrak{S}_1, \qquad R_z^l (R_z - R_z^0) \in \mathfrak{S}_1 \quad (l \geq 0). \tag{10}$$

Other variants of sufficient conditions are also possible. Thus, for example, if

$$\mathfrak{D}(\psi(H)) = \mathfrak{D}(\psi(H_0)), \tag{11}$$

where $\psi(\lambda)$ is the same as in the remark to Theorem 3, then the second of conditions (10) may be omitted. Further, the complete wave operators $W_\pm(H, H_0)$ exist if (11) is fulfilled and, for all sufficiently small $t > 0$,

$$\exp(iHt) - \exp(iH_0 t) \in \mathfrak{S}_1.$$

In conclusion we note that, in order to verify the conditions of Theorem 4 in all assertions of §§5, 6, it is convenient to use the apparatus of repeated Stieltjes operator integrals ([9], [14]).

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Note: Figure translations are in progress. See original paper for figures.