Doklady Akademii Nauk SSSR

1964

SovietRxiv

Abstract

Full Text

Doklady Akademii Nauk SSSR
1964, Vol. 156, No. 5

MATHEMATICAL PHYSICS

I. V. OLIMPIEV

ESTIMATES OF THE FIELD IN THE SHADOW REGION IN THE DIFFRACTION OF A CYLINDRICAL WAVE BY A FINITE CONVEX CYLINDER

(Presented by Academician I. G. Petrovskii, January 24, 1964)

We consider the following problem. Given the equation

$$(\Delta + k^2)U_{Q_0}(Q) = \delta(Q - Q_0) \tag{1}$$

in the exterior S_E of a closed convex curve S. We shall assume that at all points of the curve S the curvature is different from 0 and ∞ and varies sufficiently smoothly*. The point $Q_0 \in S_E$ does not belong to S.

The boundary condition is prescribed:

$$\left.\frac{\partial U}{\partial n}\right|_S = 0 \tag{2}$$

and the radiation condition

$$r^{1/2}\left(\frac{\partial U}{\partial r} - ikU\right)_{r\to\infty} \to 0. \tag{3}$$

We shall seek the asymptotics of $U_{Q_0}(Q)$ as $k \to \infty$ in the shadow region for the source Q_0. Draw from Q_0 the tangent rays to S. The arc of the curve S lying in the shadow with respect to the source Q_0 will be denoted by AmB. The shadow region is bounded by these tangent rays and the arc AmB (see Fig. 1).

In a number of works, asymptotic formulas for $U_{Q_0}(Q)$ are derived from the physical point of view (see ([1])).

At the Second All-Union Symposium on Wave Diffraction, in the report of L. A. Vainshtein, G. D. Malyuzhinets, and V. A. Fock, and later at the Copenhagen

Symposium on the Theory of Electromagnetic Oscillations, in the report of L. A. Vainshtein and V. A. Fock, the asymptotics of the solution of problem (1)–(3) was found by means of the transverse-diffusion method. However, no justification of this method has yet been given.

Theorem 1. *For $k \geq k_0$, for the solution of problem (1)–(3) in the shadow region the estimate holds*

$$|U_{Q_0}(Q)| \leq ck^{-1/2} \exp[-ck^{1/3} f(Q_0, Q)], \tag{4}$$

where $f(Q_0, Q) \geq c(D) > 0$ for any closed bounded domain D lying entirely inside the shadow region; $f(Q_0, Q)$ does not depend on k; as one approaches the boundary of light and shadow, $f(Q_0, Q) \to 0$.

In formula (4) and below, by c we denote constants depending only on the dimensions of S, whose numerical values are of no interest to us.

We outline the proof of Theorem 1. First we obtain an estimate for $U_{Q_0}(Q)$, if $Q \in AmB$. For this purpose we consider the limiting case of problem (1)–(3), when $Q_0 \in S$. If we know the solution of the problem

$$(\Delta + k^2) G_{P_0}(Q) = 0, \tag{5}$$

$$\left.\frac{\partial G}{\partial n}\right|_S = 2i\delta(P - P_0), \tag{6}$$

$$r^{1/2}\left(\frac{\partial G}{\partial r} - ikG\right) \xrightarrow[r\to\infty]{} 0, \tag{7}$$

* See condition (16); the restriction imposed by this condition on the curve S is connected with the method of proof.

where $P_0 \in S$, and $Q \in S_E$, then, applying Green's formula to the wave functions $U_{Q_0}(Q)$ and $G_{Q_0}(Q)$, we obtain

$$U_{Q_0}(P_0) = \frac{1}{2i} G_{P_0}(Q_0). \tag{7a}$$

Theorem 2. *In the shadow region for the source P_0, for $\rho(Q; S) \geqslant \delta > 0$, the estimate*

$$|G_{P_0}(Q)| \leqslant ck^{-1/2} \exp[-ck^{1/3} l(P_0, Q)], \tag{8}$$

is valid, where $l(P_0, Q)$ is the length of the shorter arc of the curve S from the point P_0 to the point of tangency of the ray drawn from Q to S.

Fig. 1

Figure 1: Fig. 1

Proof. Let

$$G_{P_0}(Q) = \Phi_{P_0}(Q) + \frac{i}{2}\int_S \Phi_P(Q) L(P_0, P)\, dl_P. \tag{9}$$

If $\Phi_P(Q)$, as a function of Q, satisfies the Helmholtz equation and

$$\left.\frac{\partial \Phi}{\partial n}\right|_S = 2i\delta(Q-P) + \left[\frac{\partial \Phi_P(Q)}{\partial n}\right], \tag{10}$$

then for the unknown function $L(P_0, P)$ we obtain the equation

$$L(P_0, Q) = 2i\left[\frac{\partial \Phi_{P_0}(Q)}{\partial n}\right] - \int_S \left[\frac{\partial \Phi_P(Q)}{\partial n}\right] L(P_0, P)\, dl_P, \tag{11}$$

which we shall solve by the method of successive approximations.

Inscribe at each point $P \in S$ a tangent circle in such a way that it lies entirely inside S. Following the work ([2]), we shall use, as the kernel $\Phi_P(Q)$, the solution of problem (5)–(7) for a circle. If Q lies outside this circle, then the following holds:

Fig. 1

Theorem 3. *The circular potential $\Phi_P(Q)$ satisfies the estimates*

$$|\Phi_P(Q)| \leqslant ck^{-1/2}\exp[-ck^{1/3} l(P, Q)], \tag{12}$$

if Q lies in the shadow with respect to the source P and $\rho(P;Q) \geqslant \delta > 0$. Here $l(P,Q)$ is the length of the shorter arc of the circle from P to the point of tangency of the ray drawn from Q to the circle,

$$|\Phi_P(Q)| \leqslant ck^{-1/2}, \tag{13}$$

if Q lies in the illuminated region and $\rho(P;Q) \geqslant \delta > 0$, and both estimates are valid up to the boundary of light and shadow; if $Q \in S$, then

$$\left.\frac{\partial \Phi}{\partial n}\right|_S = 2i\delta(Q-P) + \left[\frac{\partial \Phi_P(Q)}{\partial n}\right], \tag{14}$$

where

$$\left|\left[\frac{\partial\Phi_P(Q)}{\partial n}\right]\right| \ll \begin{cases} c\exp[-ck^{1/3}l(P,Q)] & \text{for } l(P,Q) > c\dfrac{\ln k}{k^{1/3}}, \\ c\exp[-ck^{1/3}l(P,Q)] + ck^{2/3}l(P,Q)\times \\ \quad\times\left(\dfrac{1}{r_P} - \dfrac{1}{R_{\text{cur}}(P)}\right)\exp[-ck^{1/3}l(P,Q)] & \text{for } l(P,Q) \ll c\dfrac{\ln k}{k^{1/3}}, \end{cases} \tag{15}$$

where r_P is the radius of the inscribed circle, $R_{\text{cur}}(P)$ is the radius of curvature of the curve S at the point P, and $l(P,Q)$ is the length of the smaller arc of the curve S between the points P and Q.

The proof of Theorem 3 is based on representing $\Phi_P(Q)$ in the form of a contour integral of cylindrical functions and on applying asymptotic formulas for Hankel functions to estimate this integral.

With the aid of the estimates of Theorem 3 one can without difficulty show that the Neumann series for equation (11) converges absolutely under the condition*

$$c\max_{P\in S}\left(\frac{1}{r_P} - \frac{1}{R_{\text{cur}}(P)}\right) < 1, \tag{16}$$

and its sum $L(P_0,Q)$ admits the estimate

$$|L(P_0,Q)| \ll ck^{1/3}\exp[-ck^{1/3}l(P_0,Q)]. \tag{17}$$

Now the assertion of Theorem 2 follows easily from (9), (12), (13), and (17). Thus, according to (7a) and (8), if $Q \in AmB$, then the estimate

$$|U_{Q_0}(Q)| \ll ck^{-1/2}\exp[-ck^{1/3}l(Q_0,Q)] \tag{18}$$

is valid, where $l(Q_0,Q)$ is the length of the smaller arc of the curve S from the point Q to A or B.

In order to obtain an estimate for $U_{Q_0}(Q)$ when $Q \notin S$, but is in the shadow, and thereby to complete the proof of Theorem 1, we construct the following configuration. At each point $P \in AmB$ we inscribe a circle C_P tangent to S and lying inside S. The radius of this circle may be taken independent of P. From the points A and B draw the tangents AQ_P and BQ_P to the constructed circle. We first give an estimate of the solution of problem (1)–(3) at the points of the triangle AQ_{PB} lying outside S (the shaded region in Fig. 1).

Lemma 1. Let $\overline{Q}_P$ be a point lying strictly inside the indicated region. Then

$$|U_{Q_0}(\overline{Q}_P)| \ll ck^{-1/2}\exp[-ck^{1/3}l(Q_0,\overline{Q}_P)], \tag{19}$$

where $l(Q_0, \overline{Q}_P)$ is the length of the smaller arc of S between the points of tangency to S of the tangent rays from the points $\overline{Q}_P$ and Q_0.

Let us prove this lemma. From $\overline{Q}_P$ draw the tangents $\overline{Q}_{PM}$ and $\overline{Q}_{PN}$ to C_P. Consider the auxiliary problem

$$(\Delta + k^2)\Phi_{\overline{Q}P}(Q) = \delta(Q - \overline{Q}_P) \quad \text{outside } C_P, \tag{20}$$

$$\Phi_{\overline{Q}P}(Q)\big|_{C_P} = 0, \qquad r^{1/2}\left(\frac{\partial \Phi}{\partial r} - ik\Phi\right) \longrightarrow 0, \quad r \to \infty.$$

According to (3), the solution of this problem in the shadow region, for $r_Q - r_P \geq \delta > 0$, satisfies the estimates

$$|\Phi_{\overline{Q}P}(Q)| \ll ck^{-1/2}\exp[-ck^{1/3}l(\overline{Q}_P, Q)]; \tag{21}$$

$$\left|\frac{\partial \Phi_{\overline{Q}P}(Q)}{\partial n}\right| \ll ck^{-1/2}\exp[-ck^{1/3}l(\overline{Q}_P, Q)], \tag{22}$$

* The meaning of condition (16) is that we must be able to inscribe at each point $P \in S$ a circle lying wholly

where r_Q is the distance of the point Q from the center of the circle C_P; r_P is the radius of the circle C_P; $l(\overline{Q}_P, Q)$ is the smaller arc of C_P between the points of tangency of the tangent rays to C_P from the points $\overline{Q}_P$ and Q.

Let us now apply Green' s formula to the functions $U_{Q_0}(Q)$ and $\Phi_{\overline{Q}P}(Q)$ in the exterior of S. We obtain

$$U_{Q_0}(\overline{Q}_P) = \Phi_{\overline{Q}P}(Q_0) - \int_S U_{Q_0}(P)\,\frac{\partial \Phi_{\overline{Q}P}(P)}{\partial n}\,dl_P. \tag{23}$$

To prove Lemma 1, it is enough to use (18), (21), and (22).

We shall briefly describe the subsequent constructions. We shall move the point P continuously along the arc AmB from A to B. In doing so the point Q_P describes an arc L. Let $\rho(Q; S)$ be the distance from the point Q to the curve S, i.e.

$$\rho(Q; S) = \inf_{K \in S} \rho(Q; K).$$

Let $\rho(L; S)$ be the distance between the curves L and S along the normal to S, i.e.

$$\rho(L; S) = \inf_{Q \in L}\, \inf_{K \in S} \rho(Q; K).$$

It is easily proved that

Lemma 2. $\rho(L; S) = d_0 > 0$, where L and S are the curves considered in our constructions.

It follows from this lemma that estimate (19) is valid in the shadow region between the arc AmB and its equidistant curve $S^{(1)}_{d_0}$, whose radius of curvature is greater by d_0 than the radius of curvature of AmB. Applying Lemmas 1 and 2 again, but now to $S^{(1)}_{d_0}$, we obtain that estimate (19) is valid in the shadow region lying between $S^{(1)}_{d_0}$ and some equidistant curve $S^{(2)}_{d_1}$ of it. The process can be continued analogously. Let d_j be the distance by which one may advance into the shadow from $S^{(j)}_{d_{j-1}}$ to $S^{(j+1)}_{d_j}$ in our constructions. Then we have:

Lemma 3. $\inf_j d_j = d > 0$.

This completes the proof of Theorem 1.

Similar results were obtained independently by V. M. Babich.

In conclusion I express my deep gratitude to V. A. Borovikov, who supervised this work.

Received
24 I 1964

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Note: Figure translations are in progress. See original paper for figures.