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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# INEQUALITIES FOR POSITIVE ENTIRE FUNCTIONS IN A GENERALIZED LEBESGUE SPACE

(Presented by Academician I. M. Vinogradov, 19 II 1964)

Let  $\mathcal{L}_p^{(n)}(E_n)$ , where  $\mathbf{p} = (p_1, \dots, p_n)$  and  $1 \leq p_k \leq \infty$  ( $k = 1, 2, \dots, n$ ), denote the class of functions  $f(x_1, \dots, x_n)$ , measurable in the  $n$ -dimensional Euclidean space  $E_n$ , satisfying the condition

$$\|f\|_p^{(n)} = \left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \dots \left( \int_{-\infty}^{\infty} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \dots \right]^{p_n/p_{n-1}} dx_n \right\}^{1/p_n} < \infty; \quad (1)$$

this class is called the generalized Lebesgue space (see <sup>(3)</sup>).

Denote by  $\widetilde{W}_{\vec{\sigma}}^{(\mathbf{p})}$  [ $\mathbf{p} = (p_1, \dots, p_n)$ ;  $\vec{\sigma} = (\sigma_1, \dots, \sigma_n)$ ] the class of entire functions  $f(z_1, \dots, z_n)$  of finite degree  $\vec{\sigma}$ , positive for real values of the arguments, and belonging to the generalized Lebesgue space  $\mathcal{L}_p^{(n)}(E_n)$ .

In the present note, for functions of the class  $\widetilde{W}_{\vec{\sigma}}^{(\mathbf{p})}$ , sharp inequalities of the type of S. M. Nikolskii's inequalities are obtained, which generalize and refine certain known results. In addition, a relation is established between different norms of entire functions of the class  $\widetilde{W}_{\vec{\sigma}}^{(\mathbf{p})}$  on lines parallel to the real axis.

**Theorem 1.** If  $f(z_1, \dots, z_n) \in \widetilde{W}_{\vec{\sigma}}^{(\mathbf{p})}$ , then for  $p_1 \geq p_2 \geq \dots \geq p_n$  and  $1 \leq p_k < p'_k \leq \infty$  ( $k = 1, 2, \dots, n$ ), for the function \*

$$\psi(x_n) = \left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \dots \left( \int_{-\infty}^{\infty} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{p_2/p_1} \dots \right]^{p_{n-1}/p_{n-2}} dx_{n-1} \right\}^{1/p_{n-1}}$$

we have

$$\|\psi\|_{p'_n} \leq \left(\frac{\sigma_n s_n}{2\pi}\right)^{1/p_{n-1}-1/p'_n} \|\psi\|_{p_n}, \quad (2)$$

where  $s_n = \lceil -p_n/2 \rceil$  is the least integer not less than  $p_n/2$ , and

$$\|\psi\|_q = \left(\int_{-\infty}^{\infty} |\psi(x_n)|^q dx_n\right)^{1/q}.$$

**Theorem 2.** If  $f(z_1, \dots, z_n) \in \widetilde{W}_\sigma^{(p)}$ , then for  $p_1 \geq p_2 \geq \dots \geq p_n$  and  $1 \leq p_k < p'_k \leq \infty$  ( $k = 1, 2, \dots, n$ ) we have

$$\|f\|_{p'}^{(n)} \leq \prod_{k=1}^n \left(\frac{\sigma_k s_k}{2\pi}\right)^{1/p_k-1/p'_k} \|f\|_p^{(n)}, \quad (3)$$

where  $s_k = \lceil -p_k/2 \rceil$  ( $k = 1, \dots, n$ ) is the least integer not less than  $p_k/2$ , and  $\|f\|_p^{(n)}$  is defined by equality (1).

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\* One can give an example in which the function  $\psi(x_n)$  is not an analytic function, but  $f(z_1, \dots, z_n)$  belongs to the class  $\widetilde{W}_\sigma^{(p)}$ .

We note that inequality (3) is sharp\*. In the case  $p' = \infty$  and  $p = 1$ , equality is attained for the function

$$f_0(x_1, \dots, x_n) = \prod_{i=1}^n \left(\frac{\sin \sigma_i x_i/2}{x_i}\right)^2.$$

Let us note that inequality (3) is a refinement of the corresponding inequality of I. I. Ibragimov (3) for entire functions from the class  $\widetilde{W}_\sigma^{(p)}$ .

Denote by  $B_\sigma^{(p)}$  the class of functions from the class  $W_\sigma^{(p)}$  satisfying the condition

$$|f(x + iy)| \leq |f(x - iy)| \quad (y \geq 0). \quad (4)$$

This class is a generalization of the class of functions that are real on the real axis. For this class of entire functions we give the following theorems\*\*.

**Theorem 3.** For an entire function  $f(z)$  from the class  $B_\sigma^{(p)}$ , for any  $1 \leq p \leq p' \leq \infty$ , the inequality

$$\|f(x + iy)\|_{p',x} \leq \left(\frac{s\sigma}{\pi}\right)^{1/p-1/p'} [Q_p(\sigma y) \operatorname{ch} \sigma y]^{p/p'} [Q_{p/s}(s\sigma y) \operatorname{ch} s\sigma y]^{(p'-p)/sp'} \times$$

$$\times \|f(x)\|_{p,x}, \quad (5)$$

holds, where  $s = [-p/2]$  is the smallest integer not less than  $p/2$ , and

$$Q_r(t) = \frac{\int_0^{2\pi} (1 - \sin^2 \omega \operatorname{sh}^2 t)^{r/2} d\omega}{2B(1/2r + 1/2, 1/2)}. \quad (6)$$

**Theorem 4.** For a positive entire function  $f(z)$  from the class  $B_\sigma^{(p)*}$ , for any  $1 \leq p < p' \leq \infty$ , the inequality

$$\|f(x + iy)\|_{p',x} \leq \left(\frac{s\sigma}{2\pi}\right)^{1/p-1/p'} Q_p(\sigma y) \operatorname{ch} \sigma y \|f(x)\|_{p,x}, \quad (7)$$

holds, where  $s$  and  $Q_p$  are the same as in Theorem 3.

Inequality (5) is sharp for periodic (and, consequently, trigonometric-sum) functions, if the norms are taken over  $(0, 2\pi)$  instead of  $(-\infty, \infty)$ ; in this case equality is attained for the function

$$f(z) = \cos \sigma z \quad \text{when } p' = p.$$

For entire functions of several variables  $f(z_1, \dots, z_n)$  from the classes  $W_\sigma^{(\mathbf{p})}$  and  $\widetilde{W}_\sigma^{(p)}$  we give the following theorems.

**Theorem 5.** If  $p'_1, \dots, p'_n$  are distinct numbers not less than one,  $p_1 \geq p_2 \geq \dots \geq p_n$ , and  $1 \leq p_k \leq p'_k \leq \infty$  ( $k = 1, 2, \dots, n$ ), then for an entire function from the class  $W_\sigma^{(\mathbf{p})}$  the inequality

$$\|f(x_1 + iy_1, \dots, x_n + iy_n)\|_{p'} \leq \prod_{k=1}^n \left(\frac{s_k \sigma_k}{\pi}\right)^{1/p_k - 1/p'_k} \exp\left(\sum_{k=1}^n \sigma_k |y_k|\right) \|f(x_1, \dots, x_n)\|_p \quad (8)$$

holds.

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\* Inequality (3) in the case  $p_1 = p_2 = \dots = p_n = p$ ,  $p'_1 = \dots = p'_n = p'$ , and  $n = 1$  was obtained in <sup>(1, 5)</sup>. Later a similar inequality was found by P. Boas <sup>(2)</sup> in the two-dimensional case with non-sharp constants in the case  $p_1 = p_2 = 1$  and  $p'_1 = p'_2 = \infty$ . This problem in the class of trigonometric polynomials was solved by N. Sabziev <sup>(4)</sup>.

\*\* These results were reported at the VII All-Union Conference on the Theory of Functions of a Complex Variable in September 1963 in Rostov-on-Don.

This result is obtained directly from (3, 5).

**Theorem 6.** Under the hypotheses of Theorem 5, for an entire function of the class  $\widetilde{W}_\sigma^{(\mathbf{p})}$  we have

$$\|f(x_1 + iy_1, \dots, x_n + iy_n)\|_{p'} \leq \prod_{k=1}^n \left( \frac{s_k \sigma_k}{2\pi} \right)^{1/p_k - 1/p'_k} \exp \left( \sum_{k=1}^n \sigma_k |y_k| \right) \|f(x_1, \dots, x_n)\|_p. \quad (9)$$

The collection of all entire functions of several variables  $f(z_1, \dots, z_n)$  from the class  $W_\sigma^{(\mathbf{p})}$  satisfying the condition

$$|f(x_1 + iy_1, \dots, x_n + iy_n)| \leq |f(x_1 - iy_1, \dots, x_n - iy_n)| \quad (10)$$

where  $y_1, \dots, y_n \geq 0$ , will be denoted by  $B_\sigma^{(\mathbf{p})}$ , and all positive entire functions from  $B_\sigma^{(\mathbf{p})}$  by  $\widetilde{B}_\sigma^{(\mathbf{p})}$ . For these classes of entire functions one can sharpen inequalities (8) and (9).

**Theorem 7.** Under the hypotheses of Theorem 5, for an entire function of the class  $B_\sigma^{(\mathbf{p})}$  the inequality

$$\begin{aligned} \|f(x_1 + iy_1, \dots, x_n + iy_n)\|_{p'} &\leq \prod_{k=1}^n \left( \frac{s_k \sigma_k}{\pi} \right)^{1/p_k - 1/p'_k} [Q_{p_k}(\sigma_k, y_k) \operatorname{ch} \sigma_k y_k]^{p_k/p'_k} \times \\ &\times [Q_{p_k/s_k}(s_k, \sigma_k, y_k) \operatorname{ch}(s_k \sigma_k y_k)]^{(p'_k - p_k)/s_k p'_k} \|f(x_1, \dots, x_n)\|_p, \end{aligned} \quad (11)$$

where  $s_k$  and  $Q_{p_k}$  are the same as in Theorem 3.

**Theorem 8.** Under the hypotheses and notation of the preceding theorem, for an entire function of the class  $\widetilde{B}_\sigma^{(\mathbf{p})}$  we have

$$\begin{aligned} \|f(x_1 + iy_1, \dots, x_n + iy_n)\|_{p'} &\leq \prod_{k=1}^n \left( \frac{s_k \sigma_k}{2\pi} \right)^{1/p_k - 1/p'_k} [Q_{p_k}(\sigma_k y_k) \operatorname{ch} \sigma_k y_k]^{p_k/p'_k} \times \\ &\times [Q_{p_k/s_k}(s_k, \sigma_k, y_k) \operatorname{ch} s_k \sigma_k y_k]^{(p'_k - p_k)/p'_k s_k} \|f(x_1, \dots, x_n)\|_p. \end{aligned} \quad (12)$$

Let us also note that inequalities (5) and (7) remain valid for the functions

$$\psi(x_n, y_n) = \left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \cdots \left( \int_{-\infty}^{\infty} |f(x_1, \dots, x_{n-1}, z_n)|^{p_1} dx_1 \right)^{p_2/p_1} \cdots \right]^{p_{n-1}/p_{n-2}} dx_{n-1} \right\}^{1/p_{n-1}},$$

if  $f(z_1, \dots, z_n)$  is taken respectively from  $B_{\sigma}^{(\mathbf{p})}$  or  $\widetilde{B}_{\sigma}^{(\mathbf{p})}$ .

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### CITED LITERATURE

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- <sup>4</sup> N. Sabziev, Abstracts of Reports, All-Union Conference on the Constructive Theory of Functions, Baku, 1962.
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*Note: Figure translations are in progress. See original paper for figures.*

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