SovietRxiv

Soviet-era science, translated into English

MATHEMATICS

I. B. SIMONENKO

1964

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196401.28625

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. B. SIMONENKO

ON THE MAXIMAL BOUNDARY PROPERTY OF FUNCTIONS POSSESSING INTEGRAL REPRESENTATIONS OF A CERTAIN FORM

(Presented by Academician A. N. Kolmogorov on 27 III 1964)

In the paper [1] M. Riesz proved the following theorem. Let $\varphi(z)$ be a function of the class H_p, analytic in the unit disk. At each point t of the unit circle let us place, in the same way, a sector S_t of fixed dimensions so that its vertex coincides with the point t. The dimensions of the sector are such that t is the only common point of S_t and the unit circle. Introduce the function

$$M(t) = \sup_{z \in S_t} |\varphi(z)|.$$

M. Riesz proved that $M \in L_p$ and that the estimate

$$\|M\|_{L_p} \leq c_p \|\varphi\|_{H_p},$$

holds, where c_p depends only on p. We call this property the maximal boundary property.

The present paper is devoted to a generalization of the indicated theorem of M. Riesz. The paper studies the boundary properties of functions $\varphi(P)$ having the integral representation

$$\varphi(P) = (K\psi)P = \int_\Gamma K(P,Q)\psi(Q)\,dQ, \tag{1}$$

where the kernel $K(P,Q)$ is defined for $P \in D$ and $Q \in \Gamma$, and is a continuous function on the direct product $D \times \Gamma$. Here Γ is a smooth bounded closed surface of $n-1$ dimensions, lying in n-dimensional space E_n; D is an open domain lying inside Γ.

We shall assume that the kernel K satisfies the following conditions:

1)
$$\sup_{P \in D} \left| \int_\Gamma K(P,Q)\psi(Q)dQ \right| \leq c_1 \operatorname{ess\,max} |\psi(Q)|. \tag{2}$$

SovietRxiv

2) There exists a number $m > 1$ such that for all concentric pairs of balls S and S_1 with ratio of radii $R(S) : R(S_1) \geq m = 1$ the inequality

$$|K(P,Q) - K(P,Q')| \leq c_2 \frac{|Q - Q'|}{|P - Q|^n} \tag{3}$$

holds for all $Q, Q' \in S_1 \cap \Gamma$ and P outside S, where the constant c_2 does not depend on Q, Q', S, S_1, or P.

3) If $|P - Q| \geq \gamma > 0$, then $|K(P,Q)| \leq c_3$, where c_3 depends only on γ.

We note that the kernel of the Poisson integral for the disk and for spheres of higher dimensions satisfies all these conditions. Moreover, the Green' s function of the Dirichlet problem for an elliptic equation satisfies the same conditions if the coefficients of the elliptic equation and the boundary Γ are sufficiently smooth and the maximum principle holds.

Let us prescribe a function $f(P)$ $(P \in D)$ and carry out the following construction. From an open n-dimensional ball of radius R cut out an open spherical sector of aperture $\alpha < \pi$, and place its vertex at a point $Q \in \Gamma$ so that the sector is directed inward into D and its axis of symmetry coincides with the normal to the surface Γ. We denote by S_Q the sector so placed. We shall regard R and α as chosen so that the closure of the sector S_Q, for no $Q \in \Gamma$, has any common points with Γ except the point Q. Now introduce in

consider the function

$$M_f(Q) = \sup_{P \in S_Q} |f(P)|.$$

The main result of the present paper is

Theorem 1 (on the maximal boundary property). Let $\psi \in L_p(\Gamma)$ $(p > 1)$, and let φ be connected with ψ by the integral representation (1). Then

$$\|M_\varphi(Q)\|_{L_p(\Gamma)} \leqslant c_p \|\varphi\|_{L_p\Gamma},$$

where c_p is a constant independent of ψ.

A special case of this theorem is the above-mentioned theorem of M. Riesz for analytic functions. Such a broad generalization became possible thanks to the author' s use of the ideas of Calderon and Zygmund' s work ([3]).

Theorem 1 provides a powerful means of proving the existence of angular boundary values.

Theorem 2. If the hypotheses of Theorem 1 are satisfied and the operator K maps some everywhere dense subset of $L_p(\Gamma)$ into functions continuous on the closed domain $\overline{D}$, then the function $\varphi(P)$ has limiting angular values at almost every point of Γ.

SovietRxiv

On the basis of the remark made above, it is clear that these theorems provide information about the boundary properties of solutions of the Dirichlet problem for elliptic equations.

Remark 1. Theorems 1 and 2 remain valid also in the case when Γ is an $(n-1)$-dimensional hyperplane, and D is one of the half-spaces into which Γ divides E^n.

Remark 2. The theorem carries over also to Orlicz spaces L^*_M, when M satisfies the condition

$$1 < \beta < \frac{uM'(u)}{M(u)} \leqslant \alpha$$

(for the definition of Orlicz spaces see the book ([2])).

In conclusion I wish to note that the indicated theorem of M. Riesz, generalized here, plays a very important role. Thus, it is the main support in the proof of the finest results of R. Paley and J. Littlewood ([4]) in the theory of Fourier series, which were then used by Marcinkiewicz in proving the theorem on multipliers ([5]). The significance of the latter for applications is well known.

Rostov-on-Don
State University

Received
17 X 1964

REFERENCES

1. M. Riesz, Math. Zs., **27**, 218 (1927).
2. M. A. Krasnosel' skii, Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, 1958.
3. A. P. Calderon, A. Zygmund, Acta Math., **88**, No. 1-2, 85 (1952).
4. J. Littlewood, R. Paley, Proc. London Math. Soc., **42**, No. 2, 52 (1936).
5. S. Marcinkiewicz, Studia Math., **8**, 1 (1939).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.