APPLICATION OF SEMI-RIEMANNIAN GEOMETRY TO A UNIFIED FIELD THEORY

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Abstract

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MATHEMATICS

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APPLICATION OF SEMI-RIEMANNIAN GEOMETRY TO A UNIFIED FIELD THEORY

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1°. In (1) we considered semi-Euclidean spaces. Semi-Riemannian spaces ${}^{l_1 \ldots l_r}V_n^{m_1 \ldots m_{r-1}}$ are spaces that are infinitesimally semi-Euclidean; they carry a degenerate Riemannian metric (cf. (2–4)). Here we shall study a special case of semi-Riemannian spaces ${}^{l}V_n^{m}$. For this purpose we construct a generalization of tensor calculus. We define invariants of the spaces ${}^{l}V_n^{m}$, as well as curvature flagtensors and ersatz-curvatures. The direct interpretation of ${}^{3}V_5^{4}$ gives 10 gravitational potentials g_{ik} and 4 electromagnetic ones $g_{5i}/\sqrt{g_{55}}$. The invariants are the interval, velocity, and electromagnetic action. From the field equations, which directly generalize Einstein' s equations of gravitation, one obtains Einstein' s equations and Maxwell' s equations. This solves that part of the unified-field problem which is formulated as follows: to construct, in an invariant way, a geometry that would describe gravitation and electromagnetism (5).

2°. We shall call a manifold $\mathfrak{M}_n^m$ a **flag manifold** if the following restriction is imposed on its group of admissible transformations*:

$$x'^{i}{}_{,\mu} = \partial_\mu x^i = D^i_\mu = 0$$
$$(1 \le i, j, k, l \le m < \mu, \nu, \pi \le n; \quad 1 \le \alpha, \beta \le n), \tag{1}$$

i.e., the admissible transformations in $\mathfrak{M}_n^m$ have the form

$$x^{i'} = f^i(x^1, \ldots, x^m); \qquad x^{\mu'} = f^\mu(x^1, \ldots, x^n). \tag{2}$$

A **flagtensor** $g_{\alpha\beta}$ is an object which, under (2), transforms according to the law

$$g_{i'j'} = g_{ij} D^i_{i'} D^j_{j'}, \qquad g_{\alpha'\mu'} = g_{\alpha\beta} D^\alpha_{\alpha'} D^\beta_{\mu'}. \tag{3}$$

An **object of flag-affine connection** $\Gamma^\alpha_{\beta\gamma}$ is an object transforming according to the usual law (6) and satisfying the equalities

$$\Gamma^i_{\alpha\mu} = 0 \qquad (1 \le i \le m < \mu \le n; \quad 1 \le \alpha \le n). \tag{4}$$

The **covariant derivative** of a flagtensor is the flagtensor

$$g_{ij;\alpha} = g_{ij,\alpha} - \Gamma^l_{i\alpha} g_{lj} - \Gamma^l_{j\alpha} g_{il}, \qquad g_{\beta\mu;\alpha} = g_{\beta\mu,\alpha} - \Gamma^\gamma_{\beta\alpha} g_{\gamma\mu} - \Gamma^\gamma_{\mu\alpha} g_{\beta\gamma}. \tag{5}$$

A **semi-Riemannian space** ${}^lV^m_n$ is a $\mathfrak{M}^m_n$ in which a torsion-free flag-affine connection and a symmetric flag-tensor $g_{\alpha\beta}$, satisfying the requirements:

$$\det g_{ij} \ne 0, \qquad \det g_{\mu\nu} \ne 0 \quad (1 \le i, j \le m < \mu, \nu \le n), \tag{6}$$

$$g_{ij,\mu} = 0, \tag{7}$$

$$g_{ij;k} = 0, \qquad g_{\mu\nu;\pi} = 0; \tag{8}$$

here l is an index of g_{ik}. It is easy to see that (6)–(8) are invariant with respect to (2). There exists a unique flag-tensor $g^{\alpha\beta}$, which is symmetric and is determined by the formulas

$$g_{ik} g^{kj} = \delta^j_i, \qquad g_{i\alpha} g^{\alpha\mu} = \delta^\mu_i, \qquad g_{\mu\pi} g^{\pi\nu} = \delta^\nu_\mu. \tag{9}$$

3°. We say that $\operatorname{ord} \xi^\alpha = 1$ if there exists $\xi^i \ne 0$. Otherwise $\operatorname{ord} \xi^\alpha = 2$. For a vector of the first order the expression $g_{ij}\xi^i\xi^j = |\xi|^2$, and for a vector of the second order $g_{\mu\nu}\xi^\mu\xi^\nu = |\xi|^2$, is an invariant (the square of the length of the vector). For a pair ξ and η of vectors of the same order, the expression $g_{ij}\xi^i\eta^j$ or, respectively, $g_{\mu\nu}\xi^\mu\eta^\nu$ is an invariant, called their scalar product $\xi \cdot \eta$. For vectors of different order the expressions

$$g_{ij}\xi^i\eta^j = \xi \cdot \eta \qquad (\operatorname{ord} \xi < \operatorname{ord} \eta), \tag{10}$$

$$g_{\mu\alpha}\xi^\mu\eta^\alpha = \xi \cdot \eta \qquad (\operatorname{ord} \xi > \operatorname{ord} \eta) \tag{11}$$

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[*] We denote by the symbol ${}_{,\alpha}$ ordinary differentiation with respect to x^α, and by the symbol ${}_{;\alpha}$ covariant differentiation.

are invariants, called their scalar products. The scalar product is not commutative. Vectors ξ^α, η^α are called 1-collinear if $\xi^i = \lambda \cdot \eta^i$, but $\xi^\alpha \neq \lambda \cdot \eta^\alpha$. The angle of two same-order non-1-collinear vectors is the complex number φ such that

$$\cos\varphi = \frac{\xi \cdot \eta}{|\xi| \cdot |\eta|}. \tag{12}$$

By the angle of same-order 1-collinear vectors we mean φ

$$\varphi = \frac{|\xi - \lambda\eta|}{|\xi|} = \frac{\sqrt{g_{ij} g_{\mu\nu} \begin{vmatrix} \xi^i \xi^\mu \\ \eta^i \eta^\mu \end{vmatrix} \cdot \begin{vmatrix} \xi^j \xi^\nu \\ \eta^j \eta^\nu \end{vmatrix}}}{|\xi| \cdot |\eta|}. \tag{13}$$

Formula (12) would give in this case $\cos\varphi \equiv 1$. By the angle ψ of different-order vectors, supplementary to the straight angle, we mean

$$\psi = \frac{\xi \cdot \eta}{|\xi| \cdot |\eta|}. \tag{14}$$

Thus defined, φ and ψ are additive functions of pairs of vectors. From (7) it follows that $|\xi|, |\eta|$ and φ form a complete system of invariants of the pair of vectors ξ and η. Under parallel transport $d|\xi| = 0$, $d\varphi = 0$, but $d\psi \neq 0$.

4°. We construct, as usual, the flag-tensor of curvature $R^\alpha_{\beta\gamma\delta}$. In order that in $V_n^{l,m}$ one could choose a system of coordinates in which $\Gamma^\alpha_{\beta\gamma} = 0$, it is necessary and sufficient that $R^\alpha_{\beta\gamma\delta} = 0$. But not one of the components $R_{\alpha\beta}$ is a function of $g_{i\mu}$. In order that in a domain one could choose a system of coordinates in which $g_{\alpha\beta} = \mathrm{const}$, it is necessary and sufficient that $R^\alpha_{\beta\gamma\delta} + Q^\alpha_{\beta\gamma\delta} = 0$, where

$$Q_{ijkl} = 0, \qquad Q_{\alpha\beta\gamma\mu} = g_{\mu[\alpha;\beta]\gamma}, \qquad Q_{i\mu jk} = g_{\mu[i;k]l}. \tag{15}$$

In order that at a point it be possible to choose a locally Euclidean coordinate system ($g_{\alpha\beta,\gamma} = 0$), it is necessary and sufficient that in it $g_{(\mu;\nu)} = 0$. In order that the angle between a first-order vector and a geodesic of the second order not change under parallel transport of it along the latter, it is necessary and sufficient that $g_{i(\mu;\nu)} = 0$. A space ${}^lV_n^m$ in which $g_{i(\mu;\nu)} = 0$ we call **skew**.

5°. Consider the skew ${}^3V_5^4$. Locally this space satisfies the system of postulates from (1): (Π_0^*, Π_1^*, I_2^*, I_3^*, II_4^*). Put $A_i = g_{5i} \cdot (g_{55})^{-1/2}$, $F_{ik} = 2A_{[i;k]}$. Then, without additional assumptions, in any coordinate system we obtain

$$A_{i'} = A_i D^i_{i'} + \partial_{i'} f(x^1, x^2, x^3, x^4, x^5) \qquad (1 \leq i \leq 4), \tag{16}$$

$$g_{ik,5} = 0, \qquad F_{ik,5} = 0. \tag{17}$$

Here f is an arbitrary function; (16) coincides with the gauge (gradient) transformations of the covector-potential, while (17) are the cylindricity conditions. The invariants $|dx^\alpha|$ and φ are interpreted trivially. The invariant ψ is equal to

$$\psi = A_i \frac{dx^i}{ds} + \frac{df(x^1, \dots, x^5)}{ds}.$$

This makes it possible in (8) to identify $e^{\int_M^N \psi\, ds}$ with the action of an electromagnetic field; the constant $e^{\int_M^N df}$, by which our expression differs from formula (8), § 16, is immaterial under variation. The coordinate x^5 has dimension $\frac{h}{e}$, i.e. volt-sec.

We choose the matter-electricity flag tensor $T_{\alpha\beta}$ in the form:

$$(T_{\alpha\beta}) = \begin{pmatrix} T_{ik} & a \cdot j_k \\ a \cdot j_i & 0 \end{pmatrix} \qquad (1 \le i, k \le 4;\ 1 \le \alpha, \beta \le 5), \tag{18}$$

where a is a constant, $j_k = g_{kl} j^l$, j^k is the current vector in ${}^3V_4$, and T_{ik} is the usual matter tensor.

We write the field equations in the form

$$g^{hl}(R_{lihk} + Q_{lihk}) = \varkappa \left(T_{ik} - \frac{1}{2} g_{ik} T^j_j \right), \tag{19}$$

$$g^{hl}(R_{l5h\alpha} + Q_{l5h\alpha}) = \varkappa \left(T_{5\alpha} - \frac{1}{2} g_{5\alpha} T^5_5 \right). \tag{20}$$

They are invariant. Since $T^5_5 = 0 = T_{55}$, they give the Einstein equations and the second pair of Maxwell equations (8):

$$R_{ik} = \varkappa \left(T_{ik} - \frac{1}{2} g_{ik} T \right) \qquad (1 \le i, k \le 4), \tag{21}$$

$$F^p_{k;p} = \lambda j_k, \tag{22}$$

while the first pair of Maxwell equations $F_{[ik;l]} = 0$ is obtained directly from the symmetries of $g_{\alpha\beta}$ and $\Gamma^\alpha_{\beta\gamma}$. No superfluous physically uninterpretable functions arise. If in (18) the 0 is replaced by an arbitrary scalar, then the possibility opens of introducing a new small parameter into the theory.

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Note: Figure translations are in progress. See original paper for figures.