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Abstract

Full Text

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ON THE DENSITY AND QUASI-RIEMANN HYPOTHESES

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To study the properties of zeros of L-series, Yu. V. Linnik [2] applied the principle of constructing sums that take a large value at a zero of an L-series. Here it will be shown how this principle can be supplemented by the estimates of I. M. Vinogradov [1] and a new theorem on the zeros of $\zeta(s)$ obtained. In addition, if this principle is supplemented by the estimates of Burgess [4] or A. G. Postnikov [5], then an analogous theorem can be obtained for all L-series $(\bmod D)$ having zeros near the real axis.

Theorem 1. For a given $T > 1$, all zeros of $\zeta(s)$ in the critical strip of height T can be divided into no more than $\ln T$ classes $\{C_0, C_1, \dots, C_r\}$ in such a way that the real part of the zeros of the class C_ν satisfies the estimate

$$\operatorname{Re}\rho_\nu < 1 - \eta_\nu,$$

and the total number of zeros of the class C_ν having real part $\geq \sigma$ does not exceed

$$T^{2(1+\varepsilon_\nu)(1-\sigma)} \ln^5 T,$$

where $\varepsilon_\nu = \nu/\ln T,\ \eta_\nu = C_0\varepsilon_\nu^3,\ \nu = 0, 1, \dots, r \leq \ln T$.

Proof. Let $\zeta(\rho) = 0$ and $T/2 \leq \operatorname{Im}\rho \leq T$; then

$$\sum_{n\leq T} \frac{1}{n^\rho} = O\left(T^{-\beta}\right), \qquad \beta = \operatorname{Re}\rho.$$

Multiplying the right- and left-hand sides of this equality by the sum $\sum_{n\leq T} \frac{\mu(n)}{n^\rho}$, we obtain the estimate

$$\left|\sum_{T\leq n\leq T^2} \frac{a_n}{n^\rho}\right| > \frac{1}{2}, \qquad a_n = \sum_{d|n,\ n/T\leq d\leq T} \mu(d), \qquad \operatorname{Re}\rho > \frac{1}{2}. \tag{1}$$

If we introduce the sum $S(n)=\sum_{m\le n} a_m m^{-i\gamma}$, where $\gamma=\operatorname{Im}\rho$, then by partial summation from (1) we find

$$\sum_{T\le n\le T^2}\frac{|S(n)|}{n^{1+\beta}}+\frac{|S(T)|}{T^\beta}\ge\frac{1}{2}.$$

It follows from this that, for the given zero ρ satisfying $\operatorname{Re}\rho>1/2$, $T/2\le\operatorname{Im}\rho\le T$, there exists an n_ρ from the interval (T,T^2) such that

$$|S(n_\rho)|>\frac{n_\rho^\beta}{2\ln T}. \tag{2}$$

We assign to the class C_ν those zeros for which n_ρ lies in the interval $(T_\nu,T_{\nu+1})$, where

$$T_\nu=T^{1+\varepsilon_\nu}.$$

Let $\rho\in C_\nu$. For it, the corresponding sum from (2) can be written in the form

$$S_\nu(n_\rho)=\sum_{n_\rho/T\le d\le T}\mu(d)d^{-i\gamma}\sum_{m\le n_\rho/d}m^{-i\gamma}. \tag{3}$$

I. M. Vinogradov' s method gives the estimate

$$\sum_{m\le x}m^{-i\gamma}\ll x^{1-\eta},\qquad \rho=2c_0\left(\frac{\ln x}{\ln\gamma}\right)^2. \tag{4}$$

In our case $x=n_\rho/d\ge T^{\varepsilon_\nu}$; hence $\eta\ge 2c_0\varepsilon_\nu^2$. Substituting (4) into (3), we obtain

$$|S_\nu(n_\rho)|\ll n_\rho^{1-\eta_\nu},\qquad \eta_\nu\ge c_0\varepsilon_\nu^3.$$

Comparing this inequality with the estimate (2), we find

$$\operatorname{Re}\rho_\nu\ll 1-\eta_\nu.$$

To study the density of zeros of the class C_ν, we "smooth" the sums $S_\nu(n_\rho)$:

$$S_\nu(n_\rho)=\frac{1}{2\pi i}\int_{\frac{1}{\ln T}-iT_\nu}^{\frac{1}{\ln T}+iT_\nu}\frac{(T_{\nu+1})^s}{s}\left(\sum_{m\le T_{\nu+1}}a_m m^{-i\gamma-s}\right)ds+O(1).$$

From this equality we obtain the estimate

$$|S_\nu(n_\rho)|^2 \ll \int_0^{T_\nu} \frac{\ln T}{|s|} \left| \sum_{m \le T_{\nu+1}} a_m m^{-i\gamma - s} \right|^2 dt + 1; \qquad s = \frac{1}{\ln T} + it. \quad (5)$$

But the integral on the right-hand side of the resulting inequality is not greater than

$$\sum_{0 \le n \le T_{\nu+1}} \frac{\ln T}{n+\alpha} \int_{\gamma+n}^{\gamma+n+1} \left| \sum_{m \le T_{\nu+1}} a_m m^{-it+\alpha} \right|^2 dt, \qquad \alpha = \frac{1}{\ln T}.$$

Sum the right- and left-hand sides of (5) over all ρ_ν satisfying $\operatorname{Re}\rho_\nu \ge \sigma$:

$$\sum_{\operatorname{Re}\rho_\nu \ge \sigma} |S_\nu(n_{\rho_\nu})|^2 \ll$$

$$\ll \sum_{0 \le n \le T_{\nu+1}} \frac{\ln T}{n+\alpha} \sum_{\gamma_\nu} \int_{\gamma_\nu+n}^{\gamma_\nu+n+1} \left| \sum_{m \le T_{\nu+1}} a_m m^{-it} \right|^2 dt + N_\nu\left(\sigma, T, \frac{1}{2}T\right).$$

But in a critical strip of length 1 there lie no more than $O(\ln T)$ zeros; therefore

$$\sum_{\gamma_\nu} \int_{\gamma_\nu+n}^{\gamma_\nu+n+1} \ll \ln T \cdot \int_0^{2T_{\nu+1}}.$$

Consequently,

$$\sum_{\operatorname{Re}\rho_\nu \ge \sigma} |S_\nu(n_{\rho_\nu})|^2 \ll \ln^3 T \int_0^{2T_{\nu+1}} \left| \sum_{m \le T_{\nu+1}} a_m m^{-it+\alpha} \right|^2 dt + N_\nu\left(\sigma, T, \frac{1}{2}T\right);$$

moreover, from (2) there follows the lower estimate

$$\sum_{\operatorname{Re}\rho_\nu \ge \sigma} |S_\nu(n_{\rho_\nu})|^2 > \frac{T_{\nu+1}^{2\sigma}}{4\ln^2 T} N_\nu\left(\sigma, T, \frac{1}{2}T\right).$$

Comparing the upper and lower estimates, we obtain

$$N_\nu(\sigma, T) \ll T^{2(1+\varepsilon_\nu)(1-\sigma)} \ln^5 T;$$

Theorem 1 is proved. It follows from it that the greater the density of zeros in a class, the farther they are from the unit line.

Let us consider what new information about the distribution of zeros of L-series this method gives. Let $N(\sigma, D)$ denote the number of zeros of all L-series $(\mathrm{mod}\, D)$ whose real part is $\geq \sigma$, and whose imaginary part is $\leq (\ln D)^c$. In [3], for almost all D the estimate

$$N(\sigma, D) \ll D^{2(1+1/3+\varepsilon)(1-\sigma)} \ln^{c_1} D$$

was obtained.

We shall show how this estimate can be differentiated according to classes of zeros, analogously to how this was done in Theorem 1.

Theorem 2. *For a given $D > 1$, the zeros of all L-series $(\mathrm{mod}\, D)$ satisfying $|\operatorname{Im}\rho| < \ln^c D$, $\operatorname{Re}\rho \geq 3/4$ can be divided into no more than $\ln D$ classes $(C_0, C_1, \dots, C_r)$ so that the real part of the zeros of the class C_ν has the estimate*

$$\operatorname{Re}\rho < 1 - \eta_\nu,$$

and the total number of zeros of the class C_ν having real part $\geq \sigma > 3/4$ does not exceed the quantity

$$D^{2(1+1/4+\varepsilon_\nu)(1-\sigma)} \ln^{c_1} D,$$

where

$$\varepsilon_\nu = \frac{\nu}{\ln D}, \qquad \eta_\nu = \left(\varepsilon_\nu - \frac{1}{4r} - \varepsilon\right)\frac{1}{r}, \qquad \varepsilon > 0$$

is arbitrarily small, and r is an arbitrarily large quantity.

Proof. If $L(s, \chi)$ has a zero ρ satisfying $\operatorname{Re}\rho > 3/4$, $|\operatorname{Im}\rho| < \ln^c D$, then for it, just as in the proof of Theorem 1, we obtain the inequality

$$|S(\chi, n_\rho)| > \frac{n_\rho^\beta}{\ln^{c_0} D} \qquad (D \leq n_\rho \leq D^2), \tag{6}$$

where

$$S(\chi, n_\rho) = \sum_{m \leq n_\rho} a_m \chi(m) = \sum_{n_\rho/D \leq d \leq D} \mu(d)\chi(d) \sum_{m \leq n_\rho/d} \chi(m).$$

We divide all L-series $(\mathrm{mod}\, D)$ into $O(\ln D)$ classes. To the class C_ν we assign those for which n_ρ lies in the interval $(D_\nu, D_{\nu+1})$, where $D_\nu = D^{1+\varepsilon_\nu}$.

If $\varepsilon_\nu > 1/4$, then to the sum $\sum_{m \le n_\rho/d} \chi(m)$ we apply Burgess' s estimate [4]; we obtain

$$|S_\nu(\chi, n_\rho)| \ll n_\rho^{1-\eta_\nu}, \qquad \eta_\nu \ge \left(\varepsilon_\nu - \frac{1}{4} - \frac{1}{4r} - \varepsilon\right)\frac{1}{r}.$$

Comparing this inequality with estimate (6), we find

$$\operatorname{Re}\rho_\nu < 1 - \eta_\nu.$$

In order to obtain the density of zeros in the class C_ν, we again "smooth" the sums $S_\nu(\chi, n_\rho)$, after which we obtain the estimate

$$|S_\nu(\chi, n_\rho)|^2 \ll \sum_{0 \le n \le D_{\nu+1}} \frac{\ln D}{n+\alpha} \int_n^{n+1} \left| \sum_{m \le D_{\nu+1}} \chi(m) a_m m^{-it+\alpha} \right|^2 dt, \qquad \alpha = \frac{1}{\ln D}.$$

Summing it over all characters whose L-series have zeros in this class satisfying $\operatorname{Re}\rho_\nu > \sigma$, we obtain the estimate:

$$\sum_{\chi_\nu} |S_\nu(\chi_\nu, n_\rho)|^2 \ll D_{\nu+1}^2 \ln^c D. \tag{7}$$

* The condition $\operatorname{Re}\rho > 3/4$ can in principle be replaced by $\operatorname{Re}\rho > 1/2$, at the cost of complicating the proof

But, on the other hand, from inequality (6) it follows that

$$\sum_{\chi_\nu} \left|S_\nu(\chi_\nu, \eta_\rho)\right|^2 > \frac{D_{\nu+1}^{2\sigma}}{(\ln D)^{2c_0}} N_\nu(\sigma, D). \tag{8}$$

Comparing (7) and (8), we obtain

$$N_\nu(\sigma, D) \ll D^{2(1+\varepsilon_\nu)(1-\sigma)} \ln^{c_1} D.$$

Since Burgess' s estimates are valid only for $\varepsilon_\nu > 1/4$, by combining all classes with $\nu \le \frac{1}{4}\ln D$ into one class, we obtain Theorem 2.

If one considers only moduli $D = p^n$, then the estimates of A. G. Postnikov ([5]) are applicable to the character sums

$$\sum_{m \le x} \chi(m).$$

They are nontrivial beginning with $x > D^{1/n}$. Accordingly, Theorem 2 takes the following form:

Theorem 3. *If $D = p^n$, $n \geq 4$, then the zeros of all L-series $(\bmod D)$ satisfying $|\operatorname{Im}\rho| < \ln D$ can be divided into no more than $\ln D$ classes $(C_0, C_1, \dots, C_r)$ such that the real part of the zeros of the class C_ν has the estimate*

$$\operatorname{Re}\rho_\nu < 1 - \eta_\nu,$$

and the total number of zeros of the class C_ν, having real part $\geq \sigma \geq 3/4$, does not exceed the quantity

$$D^{2(1+1/n+\varepsilon_\nu)(1-\sigma)} \ln^{c_1} D,$$

where $\varepsilon_\nu = \nu / \ln D$, $\eta_\nu = c_0 \varepsilon_\nu^3$, $\nu = 0, 1, \dots, r$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.