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Abstract

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ON THE MOTION OF A GYROSCOPIC COMPASS UNDER RANDOM VERTICAL DISPLACEMENTS OF ITS POINT OF SUPPORT

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Under conditions of vibration of the base, and in many other cases, the vertical component of the velocity of the point of support of a gyroscopic compass v_ζ may be regarded as a certain stationary random process. The equations of motion of a gyroscopic compass under vertical displacements of its point of support can be obtained from the general equations of motion of a gyrocompass given in the works ([1,2]).

Assuming, in order to simplify the analysis, that the northern and eastern components of the ship' s velocity $v_N = v_E = 0$, we obtain expressions for the components of the acceleration of the point of support of the gyrocompass along the axes of the geographically oriented coordinate system $\xi\eta\zeta$:

$$W_1 = 2v_\zeta U \cos\varphi, \qquad W_2 = RU^2 \sin\varphi\cos\varphi, \qquad W_3 = \dot{v}_\zeta - RU^2\cos^2\varphi, \quad (1)$$

where U is the angular velocity of rotation of the earth, R is the radius of the earth, and φ is the latitude of the observation site.

The components of the angular velocity of the coordinate trihedron $\xi\eta\zeta$ along its axes ξ, η, ζ will be

$$u_1 = 0, \qquad u_2 = U\cos\varphi, \qquad u_3 = \Omega = U\sin\varphi. \quad (2)$$

Restricting ourselves to the case when $W_3 \ll g$, and taking $g + W_3 \approx g$, for the conditions considered here one may represent the equations of motion of the gyroscopic compass in the form

$$dx_1/dt + (2v_\zeta/R)x_1 - \nu^2 x_2 + \Omega x_4 = 0, \qquad dx_2/dt + x_1 - \Omega x_3 = 0, \quad (3)$$

$$dx_3/dt + \Omega x_2 - (\mu^2/\nu^2)x_4 = 0, \qquad dx_4/dt - \Omega x_1 + \nu^2 x_3 = (2U\cos\varphi/R)v_\zeta,$$

where

$$x_1 = \dot{U}\cos\varphi\,\alpha, \qquad x_2 = \beta, \qquad x_3 = \gamma, \qquad x_4 = (\Xi_2/\Xi_1)U\cos\varphi\,(\delta - \delta^*), \tag{4}$$

$$\Xi_1 = 2B\cos(\varepsilon - \delta^*), \qquad \Xi_2 = 2B\sin(\varepsilon - \delta^*), \qquad \nu^2 = g/R,$$

and δ^* satisfies the equation

$$2B\sin(\varepsilon - \delta^*)U\cos\varphi = \varkappa\sin\delta^*\cos\delta^*. \tag{5}$$

Here α is the angle of rotation of the gyrosphere in azimuth; β and γ are the angles of elevation, respectively, of the northern and western diameters of the gyrosphere above the horizontal plane; δ is the angle of precession of the gyroscopes installed in the gyrosphere relative to it. The intrinsic angular momentum of each of the two gyroscopes is denoted by B; in the equilibrium position the rotor axes of the gyroscopes make an angle $\varepsilon - \delta^*$ with the northern diameter of the gyrosphere. The value of the angle δ^*, as follows from (5), depends essentially on the stiffness $\varkappa$ of the spring connecting the gyro casings with the inner surface of the gyrosphere. The quantity μ in equations (3) is close to one of the partial frequencies of the proper oscillations of the gyrocompass (under conditions when $v_\zeta = 0$) and is approximately 0.006 sec^{-1} for ordinary gyrocompasses. For gyrohorizon-compasses $\mu = \nu$ [2].

System (3) is a system of linear differential equations with random parametric excitation and a random forcing term. In view of the smallness of the quantity v_ζ/R, let us introduce explicitly into the equations of motion (3) the small parameter $\lambda = R_0/R$, where R_0 is a certain normalizing coefficient satisfying, for example, the condition $|v|_{\max}/R_0 = 1$ sec^{-1}.

We write system (3) in matrix form

$$dx/dt + ax = z(t) + y(t, x), \tag{6}$$

where

$$a = \begin{Vmatrix} 0 & -\nu^2 & 0 & \Omega \\ 1 & 0 & -\Omega & 0 \\ 0 & \Omega & 0 & -\dfrac{\mu^2}{\nu^2} \\ -\Omega & 0 & \nu^2 & 0 \end{Vmatrix}, \qquad x = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix},$$

$$y(t,x) = \left\| \begin{matrix} -\lambda \dfrac{2v_\zeta}{R_0} x_1 \\ 0 \\ 0 \\ 0 \end{matrix} \right\|, \qquad z(t) = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ \lambda \dfrac{2U\cos\varphi}{R_0} v_\zeta \end{matrix} \right\|. \tag{7}$$

From equation (6) one can pass in the usual way to the matrix integral equation

$$x(t) = N(t)x(0) + \int_0^t N(t-\tau)z(\tau)\,d\tau + \int_0^t N(t-\tau)y(\tau, x(\tau))\,d\tau, \tag{8}$$

where $N(t) = \|N_{jk}(t)\|$ $(j,k = 1,\dots,4)$ is the matrix weight function for the homogeneous matrix differential equation

$$dx/dt + ax = 0. \tag{9}$$

The characteristic equation corresponding to differential equation (9),

$$\rho^4 + (\nu^2 + \mu^2 + 2\Omega^2)\rho^2 + \nu^2\mu^2 - (\nu^2 + \mu^2)\Omega^2 + \Omega^4 = 0,$$

has two pairs of purely imaginary roots $\rho_1, \rho_2 = \pm i\omega_1$, $\rho_3, \rho_4 = \pm i\omega_2$. The functions $N_{jk}(t)$ can be represented as

$$\begin{aligned} N_{jk}(t) &= -m_{jk}\cos\omega_1 t + n_{jk}\cos\omega_2 t \quad &&\text{for } j+k = 2m, \\ N_{jk}(t) &= m_{jk}\sin\omega_1 t - n_{jk}\sin\omega_2 t &&\text{for } j+k = 2m+1, \end{aligned} \tag{10}$$

where m is a positive integer.

Introducing the matrices

$$L(t) = \|N_{j1}(t)\|, \qquad S(t) = \|N_{j4}(t)\| \quad (j = 1,\dots,4), \tag{11}$$

one can reduce integral equation (8) to the form

$$x(t) = F(t) - \lambda\frac{2}{R_0}\int_0^t L(t-\tau_1)v_\zeta(\tau_1)x_1(\tau_1)\,d\tau_1, \tag{12}$$

$$F(t) = N(t)x(0) + \lambda\frac{2U\cos\varphi}{R_0}\int_0^t S(t-\tau_1)v_\zeta(\tau_1)\,d\tau_1. \tag{13}$$

Let us rewrite (12) as a system of scalar integral equations

$$x_j(t) = F_j(t) - \lambda \frac{2}{R_0} \int_0^t N_{j1}(t-\tau_1) v_\zeta(\tau_1) x_1(\tau_1)\, d\tau_1 \quad (j = 1, \dots, 4), \tag{14}$$

$$F_j(t) = \sum_{k=1}^{4} N_{jk}(t) x_k(0) + \lambda \frac{2U\cos\varphi}{R_0} \int_0^t N_{j4}(t-\tau_1) v_\zeta(\tau_1)\, d\tau_1. \tag{15}$$

The solution of the matrix integral equation (12) can be found by the method of successive substitutions ([3]).

Substituting in the right-hand side of equation (12), instead of the function $x_1(\tau_1)$, its value from (14), and then repeatedly carrying out the analogous substitution, one can represent the desired function $x(t)$ in the form of the following absolutely and uniformly convergent series ([3]):

$$x(t) = F(t) + \sum_{n=1}^{\infty} V_n(t), \tag{16}$$

where

$$\begin{aligned} V_n(t) = (-1)^n \lambda^n \left(\frac{2}{R_0}\right)^n \int_0^t L(t-\tau_1)\, v_\zeta(\tau_1) \times \\ \times \int_0^{\tau_1} N_{11}(\tau_1-\tau_2)\, v_\zeta(\tau_2) \int_0^{\tau_2} N_{11}(\tau_2-\tau_3)\, v_\zeta(\tau_3) \dots \\ \dots \int_0^{\tau_{n-1}} N_{11}(\tau_{n-1}-\tau_n)\, v_\zeta(\tau_n)\, F_1(\tau_n)\, d\tau_n \dots d\tau_2\, d\tau_1. \end{aligned} \tag{17}$$

Substituting into (17) the expression $F_1(\tau_n)$ from (15) and retaining in the solution (16) quantities no higher than the second order of smallness, we obtain

$$\begin{aligned} x_j(t) = \sum_{k=1}^{4} x_k(0) \Bigg[N_{jk}(t) - \lambda \frac{2}{R_0} \int_0^t N_{j1}(t-\tau_1) N_{1k}(\tau_1) v_\zeta(\tau_1)\, d\tau_1 + \\ + \lambda^2 \frac{4}{R_0^2} \int_0^t \int_0^{\tau_1} N_{j1}(t-\tau_1) N_{11}(\tau_1-\tau_2) N_{1k}(\tau_2) v_\zeta(\tau_1) v_\zeta(\tau_2)\, d\tau_2\, d\tau_1 \Bigg] + \\ + \lambda \frac{2U\cos\varphi}{R_0} \int_0^t N_{j4}(t-\tau_1) v_\zeta(\tau_1)\, d\tau_1 - \\ - \lambda^2 \frac{4U\cos\varphi}{R_0^2} \int_0^t \int_0^{\tau_1} N_{j1}(t-\tau_1) N_{14}(\tau_1-\tau_2) v_\zeta(\tau_1) v_\zeta(\tau_2)\, d\tau_2\, d\tau_1 \quad (j = 1, \dots, 4). \end{aligned} \tag{18}$$

Assuming that the mean value of the random process $v_\zeta(t)$ is equal to zero, the mathematical expectation of the process $x_j(t)$ can be represented in the form

$$M[x_j(t)] = \sum_{k=1}^{4} x_k(0)\Bigg\{ N_{jk}(t) + \\ + \lambda^2 \frac{4}{R_0^2} \int_0^t N_{j1}(t-\tau_1)\left[\int_0^{\tau_1} N_{11}(\tau_1-\tau_2)N_{1k}(\tau_2)K_{vv}(\tau_1-\tau_2)\,d\tau_2\right] d\tau_1 \Bigg\} - \\ - \lambda^2 \frac{4U\cos\varphi}{R_0^2}\int_0^t N_{j1}(t-\tau_1)\left[\int_0^{\tau_1} N_{14}(\tau_1-\tau_2)K_{vv}(\tau_1-\tau_2)\,d\tau_2\right] d\tau_1, \tag{19}$$

where $K_{vv}(\tau_1-\tau_2) = M[v_\zeta(\tau_1)v_\zeta(\tau_2)]$ is the correlation function of the process $v_\zeta(t)$. By virtue of the stationarity of the process, it depends on the difference of the arguments. Denoting

$$I_k(\tau_1) = \int_0^{\tau_1} N_{11}(\tau_1-\tau_2)N_{1k}(\tau_2)K_{vv}(\tau_1-\tau_2)\,d\tau_2 \quad (k = 1, \dots, 4); \tag{20}$$

$$J(\tau_1) = \int_0^{\tau_1} N_{14}(\tau_1-\tau_2)K_{vv}(\tau_1-\tau_2)\,d\tau_2 \tag{21}$$

and replacing λ by its value, we rewrite (19) as follows:

$$M[x_j(t)] = \sum_{k=1}^{4}\left\{N_{jk}(t) + \frac{4}{R^2}\int_0^t N_{j1}(t-\tau_1)I_k(\tau_1)\,d\tau_1\right\} x_k(0) - \\ - \frac{4U\cos\varphi}{R^2}\int_0^t N_{j1}(t-\tau_1)J(\tau_1)\,d\tau_1 \quad (j = 1, \dots, 4). \tag{22}$$

Let us now determine the variances of the random processes $x_j(t)$. With zero initial conditions $x_k(0) = 0$ $(k = 1, \dots, 4)$, restricting ourselves to terms of first order of smallness, we shall have

$$x_j(t) - M[x_j(t)] = \frac{2U\cos\varphi}{R}\int_0^t N_{j4}(t-\tau_1)\,v_\zeta(\tau_1)\,d\tau_1 \qquad (j = 1, \dots, 4), \tag{23}$$

and, consequently, the variance is

$$D_j(t) = \frac{4U^2\cos^2\varphi}{R^2}\int_0^t\int_0^t N_{j4}(t-\tau_1)\,N_{j4}(t-\tau_2)\,K_{vv}(\tau_1-\tau_2)\,d\tau_2\,d\tau_1$$

$$(j = 1, \dots, 4). \tag{24}$$

As an example, consider the case when

$$K_{vv}(\tau_1 - \tau_2) = Ge^{-p|\tau_1 - \tau_2|}. \tag{25}$$

Then, according to (21),

$$J(\tau_1) = G\,[m_{14}k_1\omega_1 - n_{14}k_2\omega_2 -$$

$$- m_{14}k_1 e^{-p\tau_1}(p \sin\omega_1\tau_1 + \omega_1\cos\omega_1\tau_1) + n_{14}k_2 e^{-p\tau_1}(p\sin\omega_2\tau_1 + \omega_2\cos\omega_2\tau_1)], \tag{26}$$

where

$$m_{14} = (\Omega\omega_1^2 + \Omega\mu^2 - \Omega^3)/\omega_1(\omega_2^2 - \omega_1^2), \qquad n_{14} = (\Omega\omega_2^2 + \Omega\mu^2 - \Omega^3)/\omega_2(\omega_2^2 - \omega_1^2), \tag{27}$$

$$k_1 = (p^2 + \omega_1^2)^{-1}, \qquad k_2 = (p^2 + \omega_2^2)^{-1},$$

whence it follows that, even with zero initial conditions $x_k(0) = 0$ $(k = 1, \dots, 4)$ and zero mean values of the random disturbing forces, the mathematical expectations $M[x_j(t)]$ of the angles determining the position of the gyroscopic compass will be nonzero; i.e., on the average, under conditions of pitching and vibration, deviations arise in the gyrocompass.

The variance of the azimuthal deflection of the gyrocompass will be

$$D_1(t) = \frac{4U^2\cos^2\varphi}{R^2}\,G\,[a_0 + a_1 t + b_1\sin\omega_1 t + b_2\cos\omega_1 t + c_1\sin\omega_2 t +$$

$$+ c_2\cos\omega_2 t + d_1\sin 2\omega_1 t + d_2\cos 2\omega_1 t + f_1\sin 2\omega_2 t + f_2\cos 2\omega_2 t + \tag{28}$$

$$+ g_1\sin(\omega_1 + \omega_2)t + g_2\cos(\omega_1 + \omega_2)t + h_1\sin(\omega_1 - \omega_2)t + h_2\cos(\omega_1 - \omega_2)t],$$

where

$$a_0 = \frac{1}{2}m_{14}^2 k_1^2(3\omega_1^2 - p^2) + \frac{1}{2}n_{14}^2 k_2^2(3\omega_2^2 - p^2) - 2m_{14}n_{14}k_1k_2\omega_1\omega_2;$$

$$a_1 = p\,(m_{14}^2 k_1 + n_{14}^2 k_2); \qquad b_1 = [-2m_{14}^2 k_1^2\omega_1 p + 2m_{14}n_{14}k_1k_2\omega_2 p]e^{-pt};$$

$$b_2 = [-2m_{14}^2 k_1^2\omega_1^2 + 2m_{14}n_{14}k_1k_2\omega_2\omega_1]e^{-pt};$$

$$c_1 = [-2n_{14}^2 k_2^2\omega_2 p + 2m_{14}n_{14}k_1k_2\omega_1 p]e^{-pt};$$

$$c_2 = [-2n_{14}^2 k_2^2\omega_2^2 + 2m_{14}n_{14}k_1k_2\omega_1\omega_2]e^{-pt};$$

$$d_1 = -\frac{1}{2}m_{14}^2 k_1\omega_1^{-1}p; \qquad d_2 = \frac{1}{2}m_{14}^2 k_1; \qquad f_1 = -\frac{1}{2}n_{14}^2 k_2\omega_2^{-1}p; \qquad f_2 = \frac{1}{2}n_{14}^2 k_2;$$

$$g_1 = m_{14}n_{14}(k_1 + k_2)(\omega_1 + \omega_2)^{-1}p; \qquad g_2 = -m_{14}n_{14}k_1k_2(p^2 + \omega_1\omega_2);$$

$$h_1 = -m_{14}n_{14}(k_1 + k_2)(\omega_1 - \omega_2)^{-1}p; \qquad h_2 = m_{14}n_{14}k_1k_2(p^2 - \omega_1\omega_2).$$

It is seen from this that the variance of the gyrocompass increases with time. The growth of the variance may lead to significant errors in the readings of the gyrocompass in the absence of damping. Therefore, the interval of time during which the damping of the gyrocompass is switched off should not be too long.

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Note: Figure translations are in progress. See original paper for figures.