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Abstract

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TWO EMBEDDING THEOREMS FOR THE SPACE $L_{p,b}^{(1)}(\Omega \times R_+)$ AND THEIR APPLICATIONS

(Presented by Academician S. L. Sobolev on 14 XII 1963)

Let us denote: E_n is the n-dimensional Euclidean space of points $x = (x_1, \dots, \dots, x_n)$; E_{n-1} is the hyperplane $x_n = 0$; E_n^+ is the half-space $x_n > 0$; R_+ is the set of positive real numbers; $p = (p_1, \dots, p_n)$ is a vector for which $1 < p_i < \infty$ $(i = 1, \dots, n)$; $x' = (x_1, \dots, x_{n-1})$ is a vector of the set E_{n-1}.

Let Ω be an $(n-1)$-dimensional open set situated in E_{n-1} and having the property that for any pair of open balls K_1 and K_2 lying in Ω, there exists a finite number of open balls $K^{(1)}, K^{(2)}, \dots, \dots, K^{(m)}$ $(m \geq 2)$, also lying in Ω, such that $K^{(1)} = K_1$, $K^m = K_2$, and $K^{(i)} \cap K^{(i+1)}$ is a nonempty set for all $i = 1, \dots, m-1$. By $\Omega \times R_+$ we denote the Cartesian product of the sets Ω and R_+. Suppose that a vector $b(x) = (b_1(x), \dots, b_n(x))$ is given, whose components are defined on $\Omega \times R_+$, are measurable, positive almost everywhere, and satisfy the conditions:

1. There exists a number $R_0 \geq 0$ such that for almost all $x' \in \Omega$

$$\int_{R_0}^{\infty} b_n^{1/(1-p_n)}(x', x_n)\, dx_n < \infty. \tag{1}$$

2. There exists a nonnegative measurable function $\varphi(x_n)$, defined on (R_0, ∞), for which $\int_N^\infty \varphi(x_n)\, dx_n = \infty$ for every $N \geq R_0$, and such that

$$\int_K dx' \left[\int_{R_0}^{\infty} \varphi^{p_i'}(x_n) b_i^{1/(1-p_i)}(x', x_n)\, dx_n \right] < \infty, \tag{2}$$

where R_0 is the number from condition 1; K is an arbitrary open ball whose closure $\overline{K} \subset \Omega$, $1 \leq i \leq n-1$.

Denote by $L_{p,b}^{(1)}(\Omega \times R_+)$ the set of functions for which the inequality

$$\sum_{i=1}^{n} \left[\int_{\Omega \times R_+} b_i(x) \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right]^{1/p_i} < \infty, \tag{3}$$

holds, where $\partial u / \partial x_i$ $(i = 1, \dots, n)$ is the generalized derivative in the sense of S. L. Sobolev and $b_i(x) \geq 0$.

The following lemma is valid; it is a generalization, to spaces with a weight, of a result of S. L. Sobolev established in ([1]).

Lemma 1. For every function $u(x) \in L^{(1)}_{p,b}(\Omega \times R_+)$, where $b^{(i)}(x)$ satisfies conditions (1) and (2), there exists a constant $C(u)$ such that for

for almost all $x' \in \Omega$ the relation holds:

$$u(x', x_n) = C(u) - \int_{x_n}^{\infty} \frac{\partial u}{\partial x_n}(x', t)\, dt, \tag{4}$$

where

$$\int_{x_n}^{\infty} \frac{\partial u}{\partial x_n}(x', t)\, dt$$

converges absolutely.

Lemma 2. Let (A, C) be an interval of the real line and $-\infty \le A < C \le +\infty$. Let $f(x)$ and $b_0(x)$ be nonnegative measurable functions, with

$$\int_A^x b_0^{1/(1-p)}(t)\, dt < \infty$$

for all $x \in (A, C)$. Then the inequality

$$\left\{ \int_A^C b_0(x) \left[\frac{B_0(x)}{\left(\int_A^x B_0(t)\, dt\right)^{\frac{1}{p}\left(1+\frac{q}{p'}\right)}} \right]^p \left[\int_A^x f(t)\, dt \right]^q dx \right\}^{1/q} \le C_0 \left[\int_A^C b_0(x) f^p(x)\, dx \right]^{1/p}, \tag{5}$$

holds, where $B_0(x) = b_0^{1/(1-p)}(x)$, $1 < p \le q < \infty$,

$$C_0 = q^{-1/q} (p')^{1/q} \left(1 + \frac{q}{p'}\right)^{1/q} \left(1 + \frac{p'}{q}\right)^{1/p'}.$$

Remark. If

$$\int_x^C B_0(t)\, dt < \infty$$

for all $x \in (A, C)$, then with the same constant the inequality

$$\left\{ \int_A^C b_0(x) \left[\frac{B_0(x)}{\left(\int_x^C B_0(t)\, dt\right)^{\frac{1}{p}\left(1+\frac{q}{p'}\right)}} \right]^p \left[\int_x^C f(t)\, dt \right]^q dx \right\}^{1/q} \le C_0 \left[\int_A^C b_0(x) f^p(x)\, dx \right]^{1/p} \tag{6}$$

holds.

As a consequence of Lemmas 1 and 2, for $A = 0$, $C = \infty$, and $p = q = p_n$, we have the following embedding theorem.

Theorem 1. For every function $u(x) \in L^{(1)}_{p,b}(\Omega \times R_+)$, where $b(x)$ satisfies conditions (1) and (2), there exists a unique constant $C(u)$ such that, with a constant C_1 independent of $u(x)$, the inequality

$$\int_{\Omega\times R_+} b_n(x)\left[\frac{b_n^{1/(1-p_n)}(x)}{\int_{x_n}^{\infty} b_n^{1/(1-p_n)}(x',t)\,dt}\right]^{p_n} |u(x)-C|^{p_n}\,dx \le C_1\int_{\Omega\times R_+} b_n(x)\left|\frac{\partial u}{\partial x_n}\right|^{p_n} dx. \quad (7)$$

The following theorem is valid on boundary values of functions from the space $L^{(1)}_{p,b}(E_n^+)$ on the hyperplane E_{n-1}, without any restriction on the behavior of the weights $b_i(x)$ at infinity.

Theorem 2. Let $\Omega = E_{n-1}$, $b_i(x) = b_i(x_n)$ $(i = 1, \dots, n)$,

$$\int_0^a b_n^{-1}(t)\,dt < \infty$$

for any $a < \infty$, and $p_1 = \dots = p_n = p$. Then for every function $u(x) \in L^{(1)}_{p,b}(E_n^+)$:

1. $\lim_{x_n\to 0} u(x', x_n) = u(x', 0)$ exists and is finite for almost all $x' \in E_{n-1}$.
2. The inequality

$$\int_0^{\infty} dh\, g(h)\left[\int_{E_{n-1}} |u(x_1,\dots,x_i+h,\dots,0) - u(x_1,\dots,x_i,\dots,0)|^p\,dx'\right] \le$$

$$\le C_2\left[\int_{E_n^+} b_n(x_n)\left|\frac{\partial u}{\partial x_n}\right|^p dx + \int_{E_n^+} b_i(x_n)\left|\frac{\partial u}{\partial x_i}\right|^p dx\right], \quad (8)$$

where

$$g(h) = \min\left\{ b_n(h)\left[\frac{b_n^{1/(1-p)}(h)}{\int_0^h b_n^{1/(1-p)}(t)\,dt}\right]^p,\ b_i(h)h^{-p}\right\},$$

and C_2 is a constant independent of the function $u(x)$.

We shall indicate some applications of the results listed above to the theory of degenerate elliptic equations of the second order in the half-space E_n^+.

Consider the equation

$$Lu \equiv -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}b_{ij}(x)\frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x). \quad (9)$$

The solution of equation (9) is sought in the half-space E_n^+ under the condition

$$\lim_{x_n\to 0} u(x', x_n) = 0 \quad \text{a.e. on } E_{n-1}. \quad (10)$$

Let $\sigma(x) \geq 0$ a.e. on E_n^+. We shall say that $u(x) \in \overset{0}{L}{}^{(1)}_{2,b,\sigma}(E_n^+)$ if

$$\|u\|^2_{\overset{0}{L}{}^{(1)}_{2,b,\sigma}} = \|u\|^2_{L^{(1)}_{2,b}(E_n^+)} + \|u\|^2_{L_{2,\sigma}(E_n^+)} < \infty$$

and condition (10) is satisfied for $u(x)$.

We impose on $b(x)$ and $\sigma(x)$ the following restrictions:

a) on every ball K whose closure $\overline{K} \subset E_n^+$, for all $i = 1, \dots, n$:

$$0 < \varepsilon_1(K) \leq b_i(x) \leq \varepsilon_2(K) \quad \text{and} \quad \sigma(x) \leq \varepsilon_3(K)$$

($\varepsilon_1, \varepsilon_2, \varepsilon_3$ are constants);

b) there exist numbers $N_0 > \varepsilon_0 > 0$ such that for $x_n < \varepsilon_0$ and $x_n > N_0$, $b_n(x)$ depends only on x_n, and the conditions

$$\int_0^{\varepsilon_0} b_n^{-1}(x', t)\, dt < \infty, \qquad \int_{N_0}^{\infty} b_n^{-1}(x', t)\, dt = \infty$$

are satisfied;

c) for every pair of numbers $N > \varepsilon > 0$ there is a number $R_0(\varepsilon, N) \geq 0$ such that, for $\varepsilon < x_n < N$, for all x such that $|x'| > R_0$, we have $b_i(x) = \tilde{b}(|x'|)$ for each $i < n$ $\left(|x'|^2 = \sum_{i=1}^{n-1} x_i^2\right)$.

Introduce the function

$$\beta(x) = \sigma(x) + \frac{1}{4} b_n^{-1}(x) \left[\int_0^{x_n} b_n^{-1}(x', t)\, dt\right]^{-2}.$$

We impose on the coefficients of equation (9) the following conditions:

1) the functions $b_{ij}(x)$ and $a_i(x)$ are continuously differentiable on every ball K whose closure $\overline{K} \subset E_n^+$;

2)

$$\sum_{i,j=1}^{n} b_{ij}(x) t_i t_j \geq \sum_{i=1}^{n} b_i(x) t_i^2 \quad \text{for every } t = (t_1, \dots, t_n);$$

3) there exists a $\gamma < 1$ such that a.e. on E_n^+

$$c(x) - \sigma(x) - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} \geq -\gamma \beta(x);$$

4) there exists a constant $\tilde{c} \geq 0$ such that a.e. on E_n^+:

a) $|b_{ij}(x)|^2 \leq \tilde{c} b_i(x) b_j(x)$ $(i, j = 1, \dots, n)$;

b) $|a_i(x)|^2 \leq \tilde{c} b_i(x) \cdot \beta(x)$ $(i = 1, \dots, n)$;

c) $|c(x)| \leq \tilde{c}\beta(x)$;

5)

$$\int_{E_n^+} \beta^{-1}(x) f^2(x)\, dx < \infty.$$

Construct the bilinear functional

$$A(u,v) = \int_{E_n^+} \left[\sum_{i,j=1}^{n} b_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} a_i(x) v \frac{\partial u}{\partial x_i} + c(x) uv \right] dx,$$

which, by condition 4), is bounded in the space $\overset{0}{L}{}^{(1)}_{2,b,\sigma}(E_n^+)$. Therefore there exists a linear operator K ($\|K\| = \|A\| < \infty$) such that, identically in u and v from $\overset{0}{L}{}^{(1)}_{2,b,\sigma}(E_n^+)$, the relation

$$A(u,v) = (Ku, v)$$

holds. From the condition

$$\int_{E_n^+} \beta^{-1} f^2\, dx < \infty$$

there follows the existence of a function $\tilde{f}(x) \in \overset{0}{L}{}^{(1)}_{2,b,\sigma}(E_n^+)$ such that, identically in $v \in \overset{0}{L}{}^{(1)}_{2,b,\sigma}(E_n^+)$, the relation

$$\int_{E_n^+} f(x) v(x)\, dx = (\tilde{f}, v)$$

is satisfied.

Theorem 3. *If the coefficients and the right-hand side of equation (9) satisfy the five conditions listed above, then in the space $\overset{0}{L}{}^{(1)}_{2,b,\sigma}(E_n^+)$ equation (9) has, moreover, a unique generalized solution; furthermore, the problem of finding the generalized solution of the equation is equivalent to the problem of finding a function minimizing the quadratic functional*

$$\mathfrak{M}_f(u) = A(u, Ku) - 2A(u, \tilde{f}) = A(u, Ku) - 2 \int_{E_n^+} f(x) Ku\, dx.$$

The first boundary-value problem for a self-adjoint elliptic equation of second order in a half-space with coefficients having power-order decay at infinity was first solved by L. D. Kudryavtsev in ([2]).

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[1] S. L. Sobolev, *Siberian Mathematical Journal*, **4**, No. 3 (1963). [2] L. D. Kudryavtsev, Materials for the Joint Soviet-American Symposium on Partial Differential Equations, August, 1963, Novosibirsk.

Note: Figure translations are in progress. See original paper for figures.