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Abstract

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ON THE REDUCTION OF THE HILBERT PROBLEM FOR A MULTIPLY CONNECTED DOMAIN TO THE HILBERT PROBLEM WITH A RATIONAL COEFFICIENT

(Presented by Academician I. N. Vekua on March 26, 1964)

§ 1. Let D be a finite $(m+1)$-connected domain bounded by a contour L, consisting of the curve L_0, which encloses all the remaining curves $L_1, \dots, L_m$. The point $z=0$ lies in D. By $\omega(z, L_k)$ $(k=0,1,\dots,m)$ we denote the harmonic measure of the curve L_k with respect to the domain D, and by $\overline{\omega}(z, L_k)$ the harmonic function conjugate to it. The homogeneous Hilbert boundary-value problem (problem A_0) consists in finding a function $F(z)$, single-valued and analytic in D, H-continuous in $\overline{D}$, satisfying the boundary condition

$$\operatorname{Re}\left[\overline{\lambda(t)}\,F(t)\right] = 0 \quad \text{on } L, \tag{1}$$

where $\lambda(t) \neq 0$ and is H-continuous on L.

Problem (1) has been well studied by L. A. Kveselava ([1]), I. N. Vekua ([2]), and B. V. Boyarskii ([3,4]). The method of the regularizing multiplier is based on one of the representations:

$$\text{a)}\quad p(t)\lambda(t) = t^{\varkappa} e^{i\gamma(t)}; \qquad \text{b)}\quad p(t)\lambda(t) = t^{\varkappa} \exp\left[i \sum_{k=1}^{m} c_k \omega(t, L_k)\right] \exp[i\gamma(t)], \tag{2}$$

where $p(t)$ is a positive H-continuous function, $\varkappa = \operatorname{ind}\lambda(t)$, c_k are real constants, and $\gamma(t)$ is the boundary value of a function analytic in D, multivalued in case a) and single-valued in case b). In representation (2), instead of $t^{\varkappa}$ one may take the polynomial $(t-a_1)\cdots(t-a_{\varkappa})$, where $a_i \in D$.

If the points $a_1, \dots, a_{\varkappa}$ are regarded as variable, then in representations analogous to (2) the functions $\gamma(t)$ and $\sum_{k=1}^{m} c_k \omega(t, L_k)$ will also be variable. The question naturally arises: is it possible, by changing the points $a_1, \dots, a_\varkappa$, to ensure that in a representation of type b) the term $\exp[i \sum c_k \omega(t, L_k)]$ disappears (or, what is the same, that in a representation of type a) $\gamma(t)$ becomes the boundary value of a single-valued function)? I. N. Vekua ([2], Ch. IV, § 5) found the necessary and sufficient conditions which $a_1, \dots, a_\varkappa$ must satisfy for this. There also a solution is given of problem (1) with an integral rational coefficient, reducing essentially to the computation of Schwarz operators for the domain D.

§ 2. Without loss of generality, we shall assume that: a) $|\lambda(t)| \equiv 1$ on L; b) $\arg \lambda(t)\big|_{L_k} = 0$, $k = 1, \dots, m$, and $\frac{1}{2\pi} \arg \lambda(t)\big|_{L_0} = \varkappa$ (the index of the problem); c) the curves $L_0, \dots, L_m$ are analytic. If on L a real summable function $v(t)$ is given, then, as is known ([2,5]), the necessary and sufficient conditions for $v(t)$ to be the real part of the boundary value of a single-valued function analytic in D have the form*:

$$\int v(t)\, u_j(t)\, dt = 0, \qquad j = 1, \dots, m, \tag{3}$$

where

$$u_j(t) = \overline{t'(s)}\, \frac{\partial}{\partial n} \omega(t, L_j),$$

and $t = t(s)$ is the equation of the contour L. The functions $u_j(t)$, as is known, are boundary values of single-valued ana-

* All integrals here and below are taken over L, unless the contour is specified.

analytic in D functions $u_j(z)$ $(j = 1, \dots, m)$, whence the well-known integral representations easily follow

$$\omega(z, L_j) = -\frac{1}{2\pi} \int \ln \frac{1}{|t - z|}\, \frac{\partial \omega(t, L_j)}{\partial n}\, ds, \tag{4}$$

$$\overline{\omega}(z, L_j) = \frac{1}{2\pi} \int \arg(t - z)\, \frac{\partial \omega(t, L_j)}{\partial n}\, ds, \qquad j = 1, \dots, m. \tag{5}$$

The moduli of periodicity of the function $\overline{\omega}(z, L_j)$ are equal to the numbers

$$\int_{L_k} \frac{\partial \omega(t, L_j)}{\partial n}\, ds = \int_{L_k} u_j(t)\, dt = \int_{L_k} \omega(t, L_k) u_j(t)\, dt; \qquad k, j = 1, \dots, m. \tag{5'}$$

The functions $\overline{\omega}(z, L_j)$ may be regarded as single-valued functions $\overline{\omega}(\hat{z}, \hat{L}_j)$ on the covering surface $\hat{D}$ of the integral functions of the domain D ((6), p. 91).

On the surface $\hat{D}$, equality (5) is written in the form

$$\overline{\omega}(\hat{z}, \hat{L}_j) = \frac{1}{2\pi} \int \arg(t - \hat{z}) \, \frac{\partial \omega(t, L_j)}{\partial n} \, ds, \qquad \hat{z} \in \hat{D}, \quad j = 1, \dots, m. \qquad (6)$$

Theorem 1. For any real numbers $\delta_1 \neq 0, \dots, \delta_m \neq 0$ and $h_1, \dots, h_m$, the system of equations

$$\sum_{k=1}^{m} \delta_k \, \overline{\omega}(\hat{a}_k, \hat{L}_j) = h_j \qquad (j = 1, \dots, m) \qquad (7)$$

(where the unknowns are the points $\hat{a}_1, \dots, \hat{a}_m \in \hat{D}$) is solvable.

Proof. The left-hand side of the system (7) effects a dimension-lowering mapping of the topological product $\hat{D}^m = \hat{D} \times \dots \times \hat{D}$ into the m-dimensional Euclidean space E^m of the points $\{h_1, \dots, h_m\}$. If to each point $a_k \in \overline{D}$ one assigns some direction l_{a_k}, then one can define, uniquely in $\overline{D}^m$ (by virtue of (3)) and continuously depending on the points a_k and the directions l_{a_k}, the Jacobian of the system (7)

$$J(a_1, \dots, a_m; \, l_{a_1}, \dots, l_{a_m}) = \det_{j,k} \left\| \delta_k \frac{\partial \overline{\omega}(a_k, L_j)}{\partial l_{a_k}} \right\| = \prod_{k=1}^{m} \delta_k \, \det_{j,k} \left\| \frac{\partial \overline{\omega}(a_k, L_j)}{\partial l_{a_k}} \right\|. \quad (8)$$

We shall temporarily assume that in the system (7) the sought points satisfy the conditions $\hat{a}_1 \in \hat{L}_1, \dots, \hat{a}_m \in \hat{L}_m$, and assign to them directions tangent to L. Under these conditions the Jacobian (8) may be brought to the form:

$$\prod_{k=1}^{m} \delta_k \det_{k,j} \left\| \frac{\partial \overline{\omega}(a_k, L_j)}{ds} \right\| = \prod_{k=1}^{m} \delta_k \det_{k,j} \left\| - \frac{\partial \omega(a_k, L_j)}{\partial n} \right\| = \prod_{k=1}^{m} \delta_k t'(s_{a_k}) \, \det_{k,j} \| u_j(a_k) \|. \qquad (9)$$

The Jacobian (9) is different from zero. This follows from the properties of the functions $u_j(z)$, for any linear combination of which the assertion holds: $2N_D + N_L = 2(m-1)$, where N_D and N_L are the numbers of zeros of $f(z)$ in D and on L, respectively. The vanishing of the Jacobian (9) would contradict this assertion. In view of the continuity of the Jacobian (9) and the closedness of the set $L^m = L \times \dots \times L$, its modulus is bounded below by a positive constant.

We shall now remove the condition $\hat{a}_1 \in \hat{L}_1, \dots, \hat{a}_m \in \hat{L}_m$. Take m closed analytic nonintersecting curves $\Gamma_1, \dots, \Gamma_m$ so that the curve Γ_1 surrounds $L_1, \dots, \Gamma_m$

surrounds L_m. We seek a solution of the system (7) satisfying the conditions $\hat{a}_1 \in \hat{\Gamma}_1, \dots, \hat{a}_m \in \hat{\Gamma}_m$. The directions with respect to which the derivatives in (8) are taken are chosen tangent to $\Gamma_1, \dots, \Gamma_m$. In view of the continuous dependence of the Jacobian (8) on the points and directions and the bounded-

of the Jacobian (9), the curves Γ_k can be taken so close to L that the modulus of the Jacobian (8) will be bounded below by a positive constant, provided that $a_1 \in \Gamma_1, \dots, a_m \in \Gamma_m$. Hence it follows that (7) realizes a topological mapping of $\hat{\Gamma}^m$ into E^m. Consequently, the image of $\hat{\Gamma}^m$ under the mapping (7) is open in E^m. We shall now prove that this image is closed in E^m. To this end, introduce on the curves $\hat{\Gamma}_1, \dots, \hat{\Gamma}_m$ arc coordinates $\hat{s}_1 \leftrightarrow \hat{a}_1, \dots, \hat{s}_m \leftrightarrow \hat{a}_m$, $-\infty < \hat{s}_j < \infty$, $j = 1, \dots, m$, and denote the point $\{\hat{a}_1, \dots, \hat{a}_m\}$ by the sequence of its arc coordinates $\{\hat{s}_1, \dots, \hat{s}_m\}$. On $\hat{\Gamma}^m$ take the point $\{\hat{s}_1^0, \dots, \hat{s}_m^0\}$. The image of the unit ball S^0:

$$\sum_{k=1}^{m} |\hat{s}_k - \hat{s}_k^0|^2 < 1$$

with center at this point under the mapping (7) contains a ball in E^m. Let r be the greatest of the radii of the balls contained in the image of the ball S^0. If the point $\{\hat{s}_1^0, \dots, \hat{s}_m^0\}$ is varied continuously, then r will also vary continuously. If, during this motion, the coordinates of the center of the ball S^0 undergo a covering transformation (i.e. the points a_j^0, after traversing the contours Γ_j an integer number of times, return to their initial position), then to the left-hand sides of the equality (7) there will be added an integral linear combination of the constant moduli of periodicity (5′) of the functions $\omega(z, L_j)$. Transferring these linear combinations to the right-hand sides of (7), we see that in the system (7) only the free term has changed by a constant vector; that is, the image of the unit ball S^0 under covering transformations is not deformed, but is only translated in the space E^m by some vector. Thus r is single-valued on Γ^m and attains its positive minimum r_0. Therefore the image of the unit ball S^0 with center at any point of $\hat{\Gamma}^m$ contains a ball of radius not less than r_0. Hence the closedness of the image of $\hat{\Gamma}^m$ follows; and since it is also open, it fills all of E^m.

§ 3. We pass to the reduction of problem (1) to a problem with a rational coefficient. We seek a regularizing multiplier $p(t)$ such that the condition

$$p(t)\lambda(t) = \prod_{k=1}^{n+\varkappa}(t - a_k) \prod_{k=1}^{n}(t - b_k)^{-1} e^{u(t)+iv(t)}, \tag{10}$$

is satisfied, where $2n + \varkappa = m$ or $m + 1$, $u(z) + iv(z)$ is a single-valued analytic function in D, and a_k and b_k lie in D.

Theorem 2. *Under the stated conditions, there exist points $a_1, \dots, a_{n+\varkappa}, b_1, \dots, b_n$ in D which provide the representation (10).*

Proof. In (1), $\arg\lambda(t)$ may be assigned up to a term of the form

$$2\pi\sum_{k=1}^{m}\nu_k\omega(t, L_k),$$

where ν_k are arbitrary integers.

Connect the curves $L_1, \dots, L_m$ with the curve L_0 by nonintersecting analytic curves $l_1, \dots, l_m$, respectively. We obtain a simply connected domain D'. If $c \in D'$, while $t \in L$, then one can single out branches, single-valued on L with respect to t, of the function $\arg(t-c)$, $[\arg(t-c)]_0$. Separating the arguments in (10), we obtain:

$$\frac{1}{2\pi}[\arg\lambda(t)]_0 + \sum_{k=1}^{m}\nu_k\omega(t, L_k) =$$

$$= \frac{1}{2\pi}\sum_{k=1}^{n+\varkappa}[\arg(t-a_k)]_0 - \frac{1}{2\pi}\sum_{k=1}^{n}[\arg(t-b_k)]_0 + \frac{1}{2\pi}v(t). \tag{11}$$

We shall now require that, for $v(t)$ in equality (11), the conditions (3) be satisfied:

$$\frac{1}{2\pi}\sum_{k=1}^{n+\chi}\int[\arg(t-a_k)]_0\,u_j(t)\,dt - \frac{1}{2\pi}\sum_{k=1}^{n}\int[\arg(t-b_k)]_0\,u_j(t)\,dt = \sum_{k=1}^{m}\nu_k\int_{L_k}u_j(t)\,dt + \frac{1}{2\pi}\int[\arg\lambda]_0 u_j(t)\,dt, \qquad j \tag{12}$$

The functions on the left-hand side are, as is seen from (5), the values at the points a_k and b_k of single-valued and continuous in D' branches of the functions $\overline{\omega}(z, L_j)$, while the sum

$$\sum_{k=1}^{m}\nu_k\int_{L_k}u_j(t)\,dt$$

is an undetermined integral linear combination of the moduli of periodicity of the functions $\overline{\omega}(z, L_j)$ (this is seen from (5′)). In (12) we shall seek, instead of the points a_k and b_k, points $\hat{a}_k$ and $\hat{b}_k \in \widehat{D}$, including the sum $\sum_{k=1}^{m}\nu_k\int_{L_k}u_j(t)\,dt$ in the sums on the right-hand side. Then, denoting

$$\int[\arg\lambda(t)]_0 u_j(t)\,dt = 2\pi h_j,$$

we write (12) in the form:

$$\sum_{k=1}^{n+\chi}\overline{\omega}(\hat{a}_k, \widehat{L}_j) - \sum_{k=1}^{n}\overline{\omega}(\hat{b}_k, \widehat{L}_j) = h_j, \qquad j = 1, \dots, m. \tag{13}$$

The system (13) is a particular case of system (7). Since $2n+\chi \geq m$, the system (13) is solvable on the basis of Theorem 1. Substituting the solution of system (13) into (11), and then finding from $v(t)$ the single-valued function $u(t)$, we see that the theorem is proved. For $\chi \geq m$ one may put $n = 0$, and we obtain

Corollary 1. In a special case problem (1) may be reduced to the problem

$$\operatorname{Re}\left[(t-a_1)^{-1}\cdots(t-a_\chi)^{-1}\Phi(t)\right] = 0, \tag{14}$$

where the points $a_{m+1}, \ldots, a_\chi$ may be assigned arbitrarily, while the remaining m points may be found from the solvable system (13).

Corollary 2. In a special case problem (1) may be reduced to the problem

$$\operatorname{Re}\left[(t-b_1)\cdots(t-b_{\left[\frac{m-\chi+1}{2}\right]})(t-a_1)^{-1}\cdots(t-a_{\chi+\left[\frac{m-\chi+1}{2}\right]})^{-1}\Phi(t)\right] = 0. \tag{15}$$

The points $\hat{a}_k$ and $\hat{b}_k$ are found from (13). The solution of problems (14) and (15) can, by the method of I. N. Vekua, be reduced to the computation of a finite number of Schwarz operators, in terms of which the form of the general solutions of problems (14) and (15) is easily written.

§ 4. The proposed method makes it possible in problem (1) to remove the condition of H-continuity of $\lambda(t)$, replacing it by the condition of continuity. The solution is sought in the class H_1. In constructing $p(t)$ we dispense with the conditions of finite discontinuity and of its nonvanishing at 0. All the basic results for problem (1) remain valid in this case as well. The proposed method makes it possible to justify all results known for problem (1), independently of the theory of integral equations.

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Note: Figure translations are in progress. See original paper for figures.