

# ON THE REPRESENTATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY FOURIER SERIES IN LEGENDRE POLYNOMIALS

1964

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.26423>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**MATHEMATICS**

**P. K. SUETIN**

**ON THE REPRESENTATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY FOURIER SERIES IN LEGENDRE POLYNOMIALS**

*(Presented by Academician M. A. Lavrent'ev, May 8, 1964)*

As is known <sup>(1)</sup>, the question of expanding a function  $f(x)$ , continuous on the segment  $[-1, 1]$ , in a Fourier series in the Legendre polynomials  $\{P_n(x)\}$ , orthonormal on this segment, reduces to estimates of the uniform approximation of the function  $f(x)$  and to estimates of the Lebesgue constants as follows:

$$\left| f(x) - \sum_0^n a_k P_k(x) \right| \leq \leq |f(x) - T_n(x)| + \int_{-1}^1 |T_n(t) - f(t)| \left| \sum_0^n P_k(t) P_k(x) \right| dt, \quad x \in [-1, 1]. \quad (1)$$

In choosing polynomials  $\{T_n(x)\}$  approximating the function  $f(x)$ , it is advisable to use the following strengthening (due to A. F. Timan <sup>(2)</sup>) of Jackson's theorem:

*If the function  $f(x)$  is continuously differentiable on the segment  $[-1, 1]$   $p$  times, and  $f^{(p)}(x) \in \text{Lip } \alpha$ , then there exists a sequence of polynomials  $\{T_n(x)\}$  such that the inequality holds*

$$|f(x) - T_n(x)| \leq \frac{C_1}{n^{p+\alpha}} \left( \sqrt{1-x^2} + \frac{1}{n} \right)^{p+\alpha}, \quad x \in [-1, 1]. \quad (2)$$

This inequality can be used for the study of Fourier series in orthogonal polynomials in those cases where the singularities of the weight function do not compensate for the influence of the endpoints of the interval of orthogonality.

**Theorem 1.** *Every function  $f(x)$  satisfying on the segment  $[-1, 1]$  a Lipschitz condition of order  $\alpha > \frac{1}{2}$  is expanded in a Fourier series in Legendre polynomials which converges uniformly on this segment.*

**Proof.** The integral  $I_n(x)$  from formula (1), with the aid of estimate (2) for  $p = 0$  and by virtue of the inequality  $|a + b|^\alpha \leq C(\alpha)(|a|^\alpha + |b|^\alpha)$ , is estimated as follows:

$$\begin{aligned} I_n(x) &\leq \frac{C_1}{n^\alpha} \int_{-1}^1 \left( \sqrt{1-t^2} + \frac{1}{n} \right)^\alpha \left| \sum_0^n P_k(t) P_k(x) \right| dt \leq \\ &\leq \frac{C_2}{n^\alpha} \int_{-1}^1 (\sqrt{1-t^2})^\alpha \left| \sum_0^n P_k(t) P_k(x) \right| dt + \frac{C_3}{n^{2\alpha}} \int_{-1}^1 \left| \sum_0^n P_k(t) P_k(x) \right| dt. \end{aligned} \quad (3)$$

For the orthonormal Legendre polynomials there is the uniform estimate  $(1, 3)$

$$|P_n(x)| \leq C_4 \sqrt{n}, \quad x \in [-1, 1], \quad (4)$$

and the inequality

$$(1-t^2)^{1/4} |P_n(t)| \leq C_5, \quad t \in [-1, 1], \quad (5)$$

which is a consequence of Theorem 7.3.3 (p. 172) from  $(3)$ .

From inequality (4) we find

$$\int_{-1}^1 \left| \sum_0^n P_k(t) P_k(x) \right|^2 dt = \sum_0^n |P_k(x)|^2 \leq C_6 n^2, \quad x \in [-1, 1], \quad (6)$$

and, consequently, the second integral on the right-hand side of inequality (3) has order of growth no greater than  $n$ .<sup>\*</sup> To estimate the first integral on the right-hand side of inequality (3), denote by  $\Delta_n(x)$  that part of the segment  $[-1, 1]$  for whose points the condition  $|x-t| \leq 1/n$  is satisfied, and let  $e_n(x)$  be the remaining part of this segment. Taking (4) and (5) into account, we obtain

$$\begin{aligned} &\int_{\Delta_n(x)} (\sqrt{1-t^2})^\alpha \left| \sum_0^n P_k(t) P_k(x) \right| dt \leq \\ &\leq \int_{\Delta_n(x)} \left[ \sum_0^n (1-t^2)^{\alpha/2} |P_k(t)| |P_k(x)| \right] dt \leq \\ &\leq C_7 \frac{1}{n} \sum_0^n \sqrt{k+1} \leq C_8 \sqrt{n}, \quad x \in [-1, 1]. \end{aligned} \quad (7)$$

To estimate the integral under consideration on the set  $e_n(x)$ , we use the Christoffel-Darboux formula <sup>(1, 3)</sup>

$$\sum_0^n P_k(t)P_k(x) = \theta_n \frac{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)}{x - t}, \quad 0 < \theta_n \leq 1.$$

Since for  $t \in e_n(x)$  the inequality  $|x - t| > 1/n$  holds, with the aid of estimates (4) and (5) we find

$$\begin{aligned} & \int_{e_n(x)} (\sqrt{1-t^2})^\alpha \left| \frac{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)}{x-t} \right| dt \leq \\ & \leq C_9 \sqrt{n} \int_{e_n(x)} (\sqrt{1-t^2})^\alpha \{ |P_n(t)| + |P_{n+1}(t)| \} \frac{dt}{|x-t|} \leq \\ & \leq C_{10} \sqrt{n} \int_{e_n(x)} \frac{dt}{|x-t|} \leq C_{11} \sqrt{n} \ln n, \quad x \in [-1, 1]. \end{aligned} \quad (8)$$

Thus, the **improved Lebesgue constant** for  $x \in [-1, 1]$  admits the estimate

$$L_n^\alpha(x) = \int_{-1}^1 (\sqrt{1-t^2})^\alpha \left| \sum_0^n P_k(t)P_k(x) \right| dt \leq C_{12} \sqrt{n} \ln n, \quad (9)$$

from which the assertion of Theorem 1 follows.

**Theorem 2.** If the function  $f(x)$  is continuously differentiable on the segment  $[-1, 1]$   $p$  times and  $f^{(p)}(x) \in \text{Lip } \alpha$ , then, under the condition  $p + \alpha \geq \frac{1}{2}$ , the inequality

$$\left| f(x) - \sum_0^n a_k P_k(x) \right| \leq \frac{C_{13} \ln n}{n^{p+\alpha-1/2}}, \quad x \in [-1, 1],$$

holds.

Indeed, in deriving estimates (7) and (8), one may put the quantity  $p + \alpha \geq \frac{1}{2}$  in place of  $\alpha$ , and inequality (9) will still hold.

--

\* From formula (1) and inequality (6) it follows that every function  $f(x)$  whose first derivative is continuous on the segment  $[-1, 1]$  is expanded in a series in Legendre polynomials that converges uniformly on this segment. We note this trivial fact, which does not depend on estimate (2), because in all known to us textbooks and reference manuals the uniform convergence of the Fourier-Legendre series on the segment  $[-1, 1]$  is asserted only under the condition that

the function  $f(x)$  is twice continuously differentiable on  $[-1, 1]$ . An analogous remark also applies to Fourier series in Jacobi polynomials.

The results established hold not only for series in Legendre polynomials, but also for certain more general Fourier series in polynomials orthonormal on the segment  $[-1, 1]$  with weight  $n(x) \neq 1$ . Indeed, in the proofs of Theorems 1 and 2 only inequalities (4) and (5), and the boundedness of the constant factor on the right-hand side of the Christoffel-Darboux formula, were used; and these three conditions hold if, for example, the weight function  $n(x)$  is positive on the segment  $[-1, 1]$  and satisfies a Lipschitz condition of order  $\alpha = 1$  <sup>(3)</sup>.

The author expresses his deep gratitude to Prof. S. B. Stechkin for his attention to the present work and for a number of valuable comments.

Sverdlovsk Branch of the V. A. Steklov Mathematical Institute Academy of Sciences of the USSR

Received 27 IV 1964

## REFERENCES

- <sup>1</sup> I. P. Natanson, *Constructive Theory of Functions*, Moscow-Leningrad, 1949.
- <sup>2</sup> A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.
- <sup>3</sup> G. Szegő, *Orthogonal Polynomials*, Moscow, 1962.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*