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Abstract

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M. SHIRINBEKOV

CONSTRUCTION OF ENVELOPES OF HOLOMORPHY FOR SEMITUBE DOMAINS

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1. A (univalent) domain T over the space of complex variables $z = (z_1, \dots, z_n)$, $w = u + iv$ is called a **semitube** domain if it can be written in the form

$$T = [(z, w) : (z, u) \in B,\ |v| < \infty],$$

where B is a domain in the space of variables z, u.

A semitube domain T is called **normal** if every line $z = z^0,\ v = 0$ (z^0 is an arbitrary point of D, the projection of the domain T in the space of variables z) intersects it in a connected set.

Let (z^0, u^0) be an arbitrary point of the domain B. We assume everywhere that for the point (z^0, u^0) on the set E_0 of points $(z, u^0) \in B$ there exists a (univalent) ball σ_{z^0} with center at the point (z^0, u^0) such that $\sigma_{z^0} \subset E_0$, and if $\sigma a'_{z^0}$ is another such ball with center at the point (z^0, u^0) on E_0, then $\sigma a'_{z^0} \subset \sigma_{z^0}$.

Consider the set

$$G = [(z, u) : z \in \sigma_{z^0},\ (z, u) \in B].$$

This open set may be disconnected. By $K(B; z^0, u^0)$ we denote that connected component of the set G which contains the point (z^0, u^0). It is clear that $K(B; z^0, u^0)$ is a univalent domain (since σ_{z^0} is univalent).

A semitube domain T is called **locally normal** if, for any point $(z^0, u^0) \in B$, the domain

$$T_k = [(z, w) : (z, u) \in K(B; z^0, u^0),\ |v| < \infty]$$

is normal. It can be shown (analogously to how this was done in the work ([1])) that for every semitube domain T there exists a unique locally normal envelope T^*, and that it can be written in the form

$$T^* = [(z,w) : z \in D,\ V_1(z) < u < V_2(z),\ |v| < \infty],$$

where D is a domain without interior branch points.

2. In the work ([1]) the envelope of holomorphy $H(T)$ of a semitube domain T was constructed in the case when z is one variable. The purpose of the present note is to construct the envelope of holomorphy $H(T)$ of a semitube domain T, when $z = (z_1, z_2, \dots, z_n)$.

Namely, the following holds.

Theorem. *The envelope of holomorphy $\Gamma(T)$ of any semitube domain T can be constructed by the following two steps:*

1) by constructing the locally normal envelope T^* of the domain T;

2) by constructing the envelope of holomorphy $H(T^*)$ of the domain T^*, which coincides with the domain $\widetilde{T}$,

$$\widetilde{T} = [(z,w) : z \in H(D),\ \widetilde{V}_1(z) < u < \widetilde{V}_2(z),\ |v| < \infty],$$

where $H(D)$ is the envelope of holomorphy of D, and $V_1(z)$ and $-V_2(z)$ are the greatest pluri-

plurisubharmonic minorants, respectively, of the functions

$$R_1(z) = \begin{cases} V_1(z), & z \in D, \\ +\infty, & z \in H(D) \setminus D; \end{cases}$$

$$R_2(z) = \begin{cases} -V_2(z), & z \in D, \\ +\infty, & z \in H(D) \setminus D. \end{cases}$$

3. For the proof of the theorem formulated above, we first give several lemmas.

Lemma 1. *The functions $\widetilde{V}_1(z)$ and $-\widetilde{V}_2(z)$, defined in the theorem, always exist.*

This lemma, in the case where D and $H(D)$ are univalent, was proved in ([2]).

Lemma 2. *The semitube domain T*

$$T = [(z,w) : z \in D,\ V_1(z) < u < V_2(z),\ |v| < \infty]$$

is a domain of holomorphy if and only if: 1) D is a domain of holomorphy, 2) $V_1(z)$ and $-V_2(z)$ are plurisubharmonic functions in D.

Lemma 3. *The envelope of holomorphy $H(T)$ of any semitube domain T is a locally normal semitube domain.*

Proof. It is clear that a semitube domain possesses automorphisms of the form $z = z'$, $w' = w + it$, where t is a real parameter. Therefore $H(T)$ also possesses the indicated automorphisms (cf. ([3]), p. 224). Consequently, $H(T)$ is a semitube domain, i.e.

$$H(T) = [(z, w) : (z, u) \in B^*,\ |v| < \infty].$$

Now suppose, to the contrary, that $H(T)$ is not a locally normal semitube domain. This means that there is at least one point $(z^0, u^0) \in B^*$ for which the domain

$$T_k = [(z, w) : (z, u) \in K(B^*; z^0, u^0),\ |v| < \infty]$$

is not normal. Hence there is a point $z^* \subset \sigma_{z^0}$ such that the line $z = z^*, v = 0$ intersects T_k (or, equivalently, $K(B^*; z^0, u^0)$) not in a connected set. On the line $z = z^*, v = 0$ choose a point $(z^*, u') \in K(B^*; z^0, u^0)$ such that the interval with endpoints at (z^*, u^0) and (z^*, u') contains a point not belonging to $K(B^*; z^0, u^0)$. Without loss of generality one may assume that $u^0 < u'$. Let γ be a rectifiable curve joining, in the domain $K(B^*; z^0, u^0)$, the points (z^*, u^0), (z^*, u'), and let γ_z be its projection onto the ball σ_{z^0}, given by the equations

$$\gamma_z :\ z = z(t),\ u = u^0,$$

$$\gamma :\ z = z(t),\ u = u(t),\ 0 \le t \le 1,$$

where $z(0) = z^*$, $z(1) = z^*$, $u(0) = u^0$, $u(1) = u'$, $u(t) > u(0)$. Let $S(t)$, $0 \le t \le 1$, be the set of segments whose endpoints lie respectively on the curves γ and γ_z and are parallel to the u-axis. Since the point $(z^*, u^0) \in K(B^*; z^0, u^0)$, there exists such a t_0 that for $0 \le t < t_0$ all the segments $S(t) \subset K(B^*; z^0, u^0)$. We shall now show that there cannot exist such a first t_0, $0 < t_0 \le 1$, for which, for $0 \le t < t_0$, all the segments $S(t)$ are contained in $K(B^*; z^0, u^0)$, but the segment (interval) $S(t_0)$ contains a point not belonging to $K(B^*; z^0, u^0)$.

Suppose, to the contrary, that such a t_0 exists. Since $H(T)$ and σ_{z^0} are domains of holomorphy, it is easy to show that T_k is also a domain of holomorphy. Then $-\ln d_{T_K}(z, w)$ is a plurisubharmonic function in T_K (cf. ([4])), where $d_{T_K}(z, w)$ is the distance function in the domain T_K. It is clear that $d_{T_K}(z, w) = d_{T_K}(z, u)$. Consequently, $-\ln d_{T_K}(z, u)$ for

for each fixed $z \in \sigma_{z^0}$ is convex in the intersection of the lines $z = z'$, $v = 0$, $z' \in \sigma_{z^0}$, with the domain T_K. Hence it follows that for the segments $S(t) \subset K(B^*; z^0, u^0)$ the following maximum principle is valid:

$$-\ln d_{T_K}(z,u) = -\ln d_{T_K}(z,u)$$

$$S(t) \cup \partial S(t) \qquad\qquad \partial S(t)$$

or

$$\min_{S(t)\cup\partial S(t)} d_{T_K}(z,u) = \min_{\partial S(t)} d_{T_K}(z,u), \tag{1}$$

where $\partial S(t)$ is the boundary (the endpoints) of the segment $S(t)$. Since

$$S(t) \to S(t_0), \qquad \partial S(t) \to \partial S(t_0) \quad \text{as } t \to t_0$$

and $d_{T_K}(z,u)$ is a continuous function (cf. (4)), it follows from (1) that

$$\min_{S(t_0)\cup\partial S(t_0)} d_{T_K}(z,u) = \min_{\partial S(t_0)} d_{T_K}(z,u).$$

Since $\partial S(t_0) \subset K(B^*; z^0, u^0)$, we have

$$\min_{\partial S(t_0)} d_{T_K}(z,u) = m > 0.$$

Then

$$\min_{S(t_0)\cup\partial S(t_0)} d_{T_K}(z,u) = m > 0$$

and, consequently,

$$S(t_0) \cup \partial S(t_0) \subset K(B^*; z^0, u^0).$$

Thus we have obtained a contradiction to the fact that in the interval $S(t_0)$ there is a point not belonging to $K(B; z^0, u^0)$. At the same time Lemma 3 is proved.

4. **Proof of the theorem.** From Lemma 3 it follows that the locally normal envelope T^* of the domain T is always contained in $H(T)$, and

$$H(T^*) = \left[(z,w) : z \in D^*,\ \varphi_1(z) < u < \varphi_2(z),\ |v| < \infty\right],$$

where $D^* \subset H(D)$, $\varphi_1(z) \le V_1(z)$, $\varphi_2(z) \ge V_2(z)$, $z \in D$.

Since $H(T^*)$ is a domain of holomorphy, by Lemma 2, D^* is a domain of holomorphy and $\varphi_1(z)$, $-\varphi_2(z)$ are plurisubharmonic functions in D^*. Consequently, $D^* = H(D)$. We shall show that

$$\varphi_1(z) = \widetilde{V}_1(z), \qquad \varphi_2(z) = \widetilde{V}_2(z), \qquad z \in H(D).$$

It is clear that $\varphi_1(z) \geq \widetilde{V}_1(z)$, $\varphi_2(z) \leq \widetilde{V}_2(z)$, for $H(T^*) \subset \widetilde{T}$. On the other hand, since $\varphi_1(z) \leq R_1(z)$, $\varphi_2(z) \geq -R_2(z)$, we have $\varphi_1(z) \leq \widetilde{V}_1(z)$, $\varphi_2(z) \geq \widetilde{V}_2(z)$, $z \in H(D)$.

Consequently, $\varphi_1(z) = \widetilde{V}_1(z)$, $\varphi_2(z) = \widetilde{V}_2(z)$, and therefore $H(T^*) = \widetilde{T}$. The theorem is proved.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.