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# MATHEMATICS

A. I. VEKSLER

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**Abstract**

**Full Text**

MATHEMATICS

A. I. VEKSLER

## PARTIAL MULTIPLICATION OPERATIONS IN VECTOR STRUCTURES

*(Presented by Academician V. I. Smirnov on 24 IV 1964)*

The paper studies a  $K$ -lineal (vector structure)  $X$  with an operation of partial or full multiplication. In § 1 the principal question considered is the possibility of such a realization of the  $K$ -lineal  $X$  (as a set of continuous functions with the natural operations) under which the given multiplication would correspond to the natural multiplication of functions; in § 2—the question whether partial multiplication determines the order in  $X$ . We shall adhere mainly to the terminology of <sup>6</sup>.

§ 1. By **partial multiplication** (p.m.) in a  $K$ -lineal  $X$  is meant a partial binary operation satisfying the following conditions:

- 1) if the product  $xy$  exists, then  $yx = xy$  also exists;
- 2) if  $xy$ ,  $yz$ , and  $(xy)z$  exist, then  $x(yz) = (xy)z$  also exists;
- 3) if  $xy$  and  $xz$  exist, then  $x(y + z) = xy + xz$  also exists;
- 4) if  $\lambda$  is a real number and  $xy$  exists, then  $\lambda x \cdot y = \lambda(xy)$  also exists;
- 5) if  $x, y \geq 0$  and  $xy$  exists, then  $xy \geq 0$ .

In the present paper we shall (unless otherwise specified) consider only p.m.'s satisfying the following condition:

- 6)  $xy$  exists and is equal to 0 if and only if  $xdy$ , i.e.  $|x| \wedge |y| = 0$ .

If the product is defined for all pairs in  $X$ , we shall speak of **full multiplication** (f.m.). The class of  $K$ -lineals with f.m. is the class of commutative  $f$ -algebras in the sense of G. Birkhoff and R. S. Pierce <sup>2</sup> without nilpotent elements.

The **canonical space**  $S$  of an Archimedean  $K$ -lineal  $X$  is the carrier space of  $X$ . The Nakano <sup>10</sup>  $K$ -completion of  $X$  is a locally bicomact extremally Hausdorff space (bicomact if  $X$  contains an order unit of Freudenthal).

A p.m. in an Archimedean  $K$ -lineal  $X$  is called **realization** (r.p.m.) if it coincides with the natural multiplication of functions under some realization of  $X$  as a set of continuous functions (which may assume the values  $\pm\infty$  on nowhere dense sets) with the natural algebraic and lattice operations on the canonical space.

Previously p.m.'s (in particular, r.p.m.'s) were studied by B. Z. Vulikh (<sup>3-5</sup> and others) and G. I. Domracheva <sup>8</sup>. These authors required a p.m. to satisfy

certain additional conditions, in particular condition A:

A. If  $xy$  exists,  $|x_1| \leq |x|$ ,  $|y_1| \leq |y|$ , then  $x_1y_1$  also exists.

Condition A together with 1)–6) entails each of the conditions and :

. If  $|x_1| \leq |x|$ ,  $|y_1| \leq |y|$  and  $xy$ ,  $x_1y_1$  exist, then  $|x_1y_1| \leq |xy|$ .

. If  $xy$  exist, and also  $(z)x$ ,  $(z)y$ , and  $(z)xy$  (the projections of  $x$ ,  $y$ , and  $xy$  onto the principal component  $X_z$  generated by the element  $z$ ), then  $(z)x(z)y$  also exists (one can show that it is equal to  $(z)xy$ ).

. If  $xy$ ,  $x_1y_1$  exist and  $|x_1y_1| \leq |xy|$ , then  $(|x_1| - |x|)_+ d(|y_1| - |y|)_+$ .

.  $X_{(xy)_+} \subset X_{(x_+ \wedge y_+) \vee (x_- \wedge y_-)}$ .

Conditions A– are satisfied for any f.m. in a  $K$ -lineal, and also for any r.p.m. in a  $K$ -space. For an r.p.m. in an arbitrary Archimedean  $K$ -lineal, – are satisfied, but A may fail.

For an arbitrary p.u. a condition weaker than D is satisfied.

**Lemma 1.** For a p.u.  $X_{xy} \subset X_x \cap X_y$ . For an n.u.  $X_{xy} = X_x \cap X_y$ . For the case of a p.u. in an Archimedean  $K$ -lineal, the latter equality is contained in <sup>(10)</sup>.

**Lemma 2.** Every p.u. is extendable to a maximal one, i.e., to a non-extendable p.u. Every p.u. possessing one or several of the properties A–D is extendable to a p.u. maximal in the set of p.u.'s possessing the same properties.

Let us establish some characteristics of r.p.u.'s. We begin with the case of n.u.

**Theorem 1.** An n.u. in an Archimedean  $K$ -lineal  $X$  is realizational.

In essence the theorem is not new. For a  $K$ -space it was proved by H. Nakano <sup>(10)</sup>. For the general case, I. Amemiya <sup>(1)</sup>, D. Johnson <sup>(7)</sup>, and D. Kist <sup>(9)</sup> proved the possibility of realizing  $X$  in the form of an algebra of continuous functions with the natural operations on spaces which are, generally speaking, different from the canonical space  $S$ . To every realization of an Archimedean  $K$ -lineal with an n.u. on an arbitrary topological space  $T$  there corresponds a realization on  $S$ . However, from the possibility of realizing some n.u. on  $T$  it does not follow that every r.p.u. can be realized on the same  $T$ . Thus, even for a  $K_\sigma$ -space  $X$  with a unit, not every r.p.u. corresponds to a realization on the quasi-extremal bicompart—its own space of  $X$  itself.

**Remark.** The assertion (converse to Theorem 1) that every r.p.u. is complete holds if and only if every principal component in  $X$  is an extended  $K$ -space.

We pass to proper p.u.'s. Suppose first that there is a **multiplicative unit** (m.u.), i.e., such an  $e \in X$  that for any  $x \in X$  there exists  $xe = x$ . By virtue of b), the m.u. is also an order unit (Freudenthal).

**Theorem 2.** In an Archimedean  $K$ -lineal, a p.u. with m.u. is realizational if and only if it is maximal in the set of all p.u.'s satisfying B.

The absence of an m.u. considerably complicates matters. In particular, there may exist almost empty r.p.u.'s; a p.u. is called **almost empty** if multiplication is defined only for disjoint pairs as in b). One almost empty r.p.u. is obtained knowingly from many realizations; every almost empty r.p.u. can be extended in many ways to other r.p.u.'s. An arbitrary r.p.u. turns out to be non-extendable to another r.p.u. if and only if the set  $E$  of all products  $xy$  is complete, i.e., from  $zdE$  it follows that  $z = 0$ . We shall call a p.u. possessing the last property **sufficiently broad**. A sufficiently broad r.p.u. is obtained, in particular, from one realization. In a  $K$ -space there are no r.p.u.'s distinct from sufficiently broad ones. We give a characteristic of sufficiently broad r.p.u.'s for a  $K$ -lineal with projections onto the principal component (p.g.k.  $K$ -lineal).

**Theorem 3.** *In a p.g.k. Archimedean  $K$ -lineal, a sufficiently broad p.u. is realizational if and only if it is maximal in the set of all p.u.'s satisfying  $B$  and  $C$ .*

**Remark 1.** In the formulation of Theorem 3 one may also consider condition  $B$  only for  $x, y > 0$ .

**Remark 2.** In the case of a  $K$ -space one may additionally assert only that every r.p.u. is a maximal p.u. At the same time there exist p.g.k. Archimedean  $K$ -lineals in which no r.p.u. is a maximal p.u.

In the case of an arbitrary Archimedean  $K$ -lineal  $X$ , even a sufficiently broad r.p.u. may be too poor. For example, it may happen that in an infinite-dimensional  $K$ -lineal the product exists only for disjoint pairs, and also for pairs of the form  $\alpha x_0 \cdot \beta x_0$ , where  $x_0$  is a fixed element. Such a p.u., as a rule, can be extended while preserving very strong properties inherent in r.p.u.'s. In view of this, it seems scarcely possible to us to obtain in the general case a good characteristic of r.p.u.'s,

similar to the characteristics obtained in Theorems 2 and 3. Let us consider one special case—the case of a p.m. with a complete system of partial units of multiplication. A positive element is called a **partial unit of multiplication** (p.u.m.) if it is a unit of multiplication for the component generated by it.

**Theorem 4.** *A p.m. with a complete system of p.u.m. in an Archimedean  $K$ -linear lattice is realizational if and only if it is maximal in the set of all p.m.'s satisfying  $B$ ,  $\Gamma$ , and .*

**Remark.** In the absence of a complete system of p.u.m., the theorem ceases to be true even for a  $K$ -space (as regards sufficiency).

**Example 1.** Let  $X$  be the  $K$ -space of all pairs  $(a, b)$  of real numbers with the natural linear operations and coordinatewise ordering. In  $X$  there are exactly  $c$  distinct r.p.m.'s (all of them complete), each completely determined by prescribing the products  $(1, 0)^2 = (\alpha, 0) > 0$ ,  $(0, 1)^2 = (0, \beta) > 0$ .

In  $X$  there are  $2^c$  maximal p.m.'s. Let  $V$  be any collection of pairs  $[\lambda, \mu]$  of real numbers such that  $\lambda, \mu \neq 0$ , and each real  $\delta \neq 0$  occurs in exactly one pair

( $\delta = \lambda \neq \mu$ ,  $\delta = \mu \neq \lambda$ , or  $\delta = \lambda = \mu$ ). Consider the following p.m. The product exists for disjoint pairs, and also for pairs of elements of the form

$$\alpha(1, \lambda) \cdot \beta(1, \mu) = \alpha\beta x_{\lambda\mu} \quad ([\lambda, \mu] \in V; \alpha, \beta \in (-\infty, +\infty)),$$

where  $x_{\lambda\mu} \in X$ , with  $x_{\lambda\mu} \neq 0$  and  $x_{\lambda\mu} > 0$  for  $\lambda, \mu > 0$ . This p.m. is either a part of a realizational one, or is maximal. Conversely, every maximal p.m. can be obtained in this way.

If  $x_{\lambda\mu} = (1, 0)$ , then one obtains a maximal p.m. which is not sufficiently wide. For it the equality considered in Lemma 1 is not satisfied.

§ 2. We shall say that a p.m. in an arbitrary  $K$ -linear lattice  $X$  **defines an order** in  $X$  if the linear set with p.m.  $X$  cannot, under another order, be made into a  $K$ -linear lattice (we note that if, for example, under the first order  $X$  is a  $K$ -space, then under the second order even fulfillment of the Archimedean principle with the same p.m. is not required). Obviously, a p.m. defines an order in  $X$  if and only if every algebraic isomorphism of  $X$  onto a  $K$ -linear lattice with p.m.  $Y$  is also an order isomorphism.

It is clear that an almost empty p.m. does not define an order in a  $K$ -linear lattice (for example, the order can be changed to the opposite one). On the other hand, it is well known that the order in the algebra of real numbers is defined uniquely.

**Theorem 5.** *In a  $K$ -space every r.p.m. (in particular, every p.m.) defines an order. In a  $K_\sigma$ -space every p.m. defines an order.*

Earlier G. Birkhoff and R. Pierce <sup>(2)</sup> proved a weaker assertion. They showed that every algebraic isomorphism between two  $K$ -spaces ( $K_\sigma$ -spaces) with p.m. and u.m. is also a lattice isomorphism.

**Remark.** The question remains open whether an r.p.m. defines an order in a  $K_\sigma$ -space (recall that a realization on the extremal space  $S$  is being considered).

**Example 2.** Let  $X$  (and  $Y$ ) be an Archimedean p.g.c.  $K$ -linear lattice with the natural order and p.m. with u.m., consisting of functions on  $[1, \infty)$ , piecewise continuous of the following form. For every  $x \in X$  (respectively  $y \in Y$ ) there are  $1 = a_0 < a_1 < \dots < a_n = \infty$  and real  $\alpha_{ik}$  ( $k = 0, 1, \dots, r_i$ ;  $i = 1, 2, \dots, n$ ) such that, for  $t \in [a_{i-1}, a_i)$ ,

$$x(t) = \sum_{k=0}^{r_i} \alpha_{ik} t^k \quad \left( y(t) = \sum_{k=0}^{r_i} \alpha_{ik} t^{-k} \right).$$

The natural correspondence  $x \leftrightarrow y$  is an isomorphism between the algebras  $X$  and  $Y$ . However, this isomorphism is not structural (incidentally, in  $Y$ , unlike in  $X$ , there is a strong unit). It is interesting to note that in  $X$  one can also define

a non-Archimedean order with the same p.m. (if  $x > 0$  is taken to mean the case in which the leading nonzero coefficients  $\alpha_{ir_i}$  are positive, in each interval).

Thus, Theorem 5 is not true in the general case even for an Archimedean p.o.r.  $K$ -lineal (and in fact even for an Archimedean  $K$ -lineal with projections onto any component) with p.m. and u.m.

In a discrete  $K$ -lineal an r.p.m. always determines an order. An arbitrary maximal p.m., even in a  $K$ -space, need not determine an order (see Example 1).

Let us also observe that if, in defining  $X$ , one starts, for example, from sums of the form

$$\sum_{k=0}^{r_i} \alpha_{ik} t^{k+\pi},$$

then the corresponding realization of  $X$  on  $S$  gives an almost empty r.p.m.

One may also consider  $l$ -groups (additive structures) with p.m. Thus, for example, it turns out that a sufficiently broad r.p.m. (in particular, a p.m.) determines an order in a complete  $l$ -group.

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Leningrad Textile Institute  
named after S. M. Kirov

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*Note: Figure translations are in progress. See original paper for figures.*

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