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Abstract

Full Text

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ON ABSOLUTE SUMMABILITY BY THE METHODS OF CESÀRO, RIESZ, AND ZYGMUND

(Presented by Academician P. S. Novikov, 3 XII 1963)

§ 1. Let a certain summability method T be given by the matrix $\|a_{nk}\|$. The series

$$\sum_{k=0}^{\infty} u_k \tag{1}$$

is called **absolutely summable by the method T** ($|T|$ -summable) if the σ_n have meaning,

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} S_k, \quad \text{where } S_k = \sum_{i=0}^k u_i,$$

and the series

$$\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|$$

converges.

In this note we present some results concerning absolute summability of orthogonal and trigonometric series by Cesàro methods of various orders and by the methods of Zygmund and Riesz. The Cesàro means ((C, α) -means) of order α ($\alpha > -1$) of the series (1) are defined by the formula

$$\sigma_n^{(\alpha)} = \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^n A_{n-k}^{(\alpha)} u_k = \frac{1}{A_n^{(\alpha)}} \left[\sum_{k=0}^{n-1} (A_{n-k}^{(\alpha)} - A_{n-k-1}^{(\alpha)}) S_k + A_0^{(\alpha)} S_n \right],$$

where

$$A_n^{(\alpha)} = (\alpha + 1) \cdots (\alpha + n) / n!.$$

The normal Zygmund means of order α ($\alpha > 0$) ((Z, α) -means) are given by the formula

$$\sigma_n^{(\alpha)} = \sum_{k=0}^n \left[1 - \left(\frac{k}{n+1} \right)^\alpha \right] u_k = \frac{1}{(n+1)^\alpha} \sum_{k=0}^n [(k+1)^\alpha - k^\alpha] S_k.$$

Let $\{\lambda_n\}$ be a sequence strictly increasing to infinity. The Riesz means $((R, \lambda_n)$ -means) corresponding to this sequence are computed by the formula

$$\sigma_n(\lambda) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_{n+1}} \right) u_k = \sum_{k=0}^n \frac{\lambda_{k+1} - \lambda_k}{\lambda_{n+1}} S_k.$$

Tandori ⁽⁷⁾ obtained the following criterion for $|C, 1|$ -summability of orthogonal series.

For all orthogonal series

$$\sum_{k=0}^{\infty} a_k \varphi_k(x) \quad (x \in [0, 1], \{a_k\} \text{ fixed}) \quad (2)$$

to be $|C, 1|$ -summable almost everywhere on $[0, 1]$, it is necessary and sufficient that

$$\sum_{m=1}^{\infty} A_m < \infty, \quad \text{where } A_m = \left(\sum_{k=2^{m+1}}^{2^{m+1}} a_k^2 \right)^{1/2}. \quad (3)$$

Billard ⁽²⁾ showed that the necessity of this condition is realized on the Rademacher system. Leindler ⁽³⁾ obtained conditions of the same type, necessary and sufficient for $|C, \alpha|$ -summability almost everywhere of orthogonal series for various α . By Moricz ⁽⁴⁾, and also by Aleksich and Kralik ⁽¹⁾, Tandori' s result was generalized in another direction—to absolute summation by Riesz methods. Aleksich and Kralik ⁽¹⁾ also obtained a certain test for $|R, \lambda_n|$ -summability of orthogonal series under the lacunarity condition on the series, if the coefficients of the series can be majorized by a positive sequence decreasing monotonically to zero. In the case when $\lambda_n = n$, a test of this kind had earlier been obtained by Leindler.

Of interest are tests for absolute summation of series expressed in terms of Weyl multipliers. P. L. Ul' yanov ⁽⁹⁾ proved the theorem:

If $\omega(n), \omega(n) \uparrow \infty$, is such that

$$\sum_{n=1}^{\infty} \frac{1}{n\omega(n)} < \infty, \quad (4)$$

then $\omega(n)$ is a Weyl multiplier for $|C, 1|$ -summability almost everywhere of orthogonal series. If, however, $\omega(n) \uparrow \infty$ is such that (4) does not hold, then $\omega(n)$ is not a Weyl multiplier for $|C, 1|$ -summability almost everywhere of trigonometric series.

Results of the same kind are also given for the methods $|C, \alpha|$.

The aim of the present paper is to give such tests for the absolute summability of series as would make it possible to judge, from the structural properties of a function, the absolute summability of its Fourier series. These tests are analogous to those given by S. B. Stechkin^(5, 6) for absolute convergence. For $|C, \alpha|$ -summability with $0 < \alpha < 1/2$, a question of this kind was studied by M. F. Timan⁽⁸⁾.

§ 2. In this section we present some results concerning $|C, \alpha|$ -summability and $|Z, \alpha|$ -summability of orthogonal series.

Theorem 1. 1) In order that the series (2) with respect to an arbitrary orthonormal system of functions be $|C, \alpha|$ -summable almost everywhere on $[0, 1]$, it is sufficient: in the case $\alpha > 1/2$, that the condition

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\lg n}} E_n < \infty, \quad (5)$$

hold; in the case $\alpha = 1/2$, that the condition

$$\sum_{n=1}^{\infty} \frac{1}{n} E_n < \infty, \quad (6)$$

hold; in the case $-1 < \alpha < 1/2$, that the condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^{1+2\alpha}}} E_n < \infty, \quad (7)$$

hold, where

$$E_n = \left(\sum_{k=n}^{\infty} a_k^2 \right)^{1/2}.$$

- 2) If $A_n \downarrow 0$, then condition (5), and if $a_n \downarrow 0$ and $A_n \downarrow$, then condition (6) are necessary for $|C, \alpha|$ -summability almost everywhere of all orthogonal series for $\alpha > 1/2$ and $\alpha = 1/2$, respectively. If $-1 < \alpha < 1/2$ and $a_n \downarrow 0$, then condition (7) is necessary for $|C, \alpha|$ -summability almost everywhere of all orthogonal series of the form (2).

- 3) The condition $A_n \downarrow 0$ in the preceding item is essential and cannot be replaced by the condition $a_n \downarrow 0$.

The necessity in this theorem is realized on the Rademacher system. Moreover, if, for example, condition (5) is not satisfied and $A_n \downarrow 0$, then the su-

there exists a trigonometric series with coefficients b_k , almost everywhere not $|C, \alpha|$ -summable for any $\alpha > 1/2$, and such that

$$E_n = \left(\sum_{k=n}^{\infty} b_k^2 \right)^{1/2}.$$

Analogous results are also valid for the remaining α^* . It turns out that sufficient conditions for $|C, \alpha|$ -summability in the form just formulated for $1/2 \leq \alpha < \infty$ are equivalent to the corresponding conditions expressed in terms of Weyl multipliers. If, however, $-1 < \alpha < 1/2$, then the sufficient conditions of Theorem 1 are less restrictive than the conditions expressed in terms of Weyl multipliers.

For $|Z, \alpha|$ ($\alpha > 0$)-summability almost everywhere of orthogonal series, the necessary and sufficient conditions coincide exactly with the necessary and sufficient conditions obtained by Tandori for $|C, 1|$ -summability, i.e. for $|Z, \alpha|$ -summability almost everywhere of orthogonal series it is necessary and sufficient that condition (3) be fulfilled. In exactly the same way, condition (5) will be sufficient, and, when $A_n \downarrow 0$, also necessary, for $|Z, \alpha|$ -summability of orthogonal series. The criterion using Weyl multipliers also coincides verbatim with the corresponding criterion for $|C, 1|$ -summability.

§ 3. In this section we present some results concerning absolute summation by Riesz methods.

Let $\{\lambda_n\}$ be a convex sequence increasing from 2 to ∞ , and let $\lambda(x)$ be a convex, differentiable, strictly increasing function such that $\lambda(n) = \lambda_n$.

Theorem 2. *Let $\{\lambda_n\}$ be either a) a convex upward sequence, or b) a convex downward sequence such that $\lambda(x+1)/\lambda(x) < q$, starting from some x for some $q > 1$, and $\lambda'(x)/\lambda(x)$ decreases monotonically.*

- 1) *The condition*

$$\sum_{n=1}^{\infty} \frac{\lambda'(n)E_n}{\lambda(n)\sqrt{\lg \lambda(n)}} < \infty \tag{8}$$

is sufficient for $|R, \lambda_n|$ -summability almost everywhere of orthogonal series of the form (2).

- 2) *Let*

$$A_n(\lambda) = \left\{ \sum_{k=[l(2^n)]+1}^{[l(2^{n+1})]} a_k^2 \right\}^{1/2}, \quad \text{where } l(x) = \lambda^{-1}(x).$$

Then, if $A_n(\lambda) \downarrow 0$, condition (8) is necessary for $|R, \lambda_n|$ -summability almost everywhere of all orthogonal series of the form (2).

- 3) For convex upward sequences $\{\lambda_n\}$, the requirement $A_n(\lambda) \downarrow 0$ in item 2) is essential and cannot be replaced by the requirement $a_n \downarrow 0$. For convex downward sequences, the condition $A_n(\lambda) \downarrow 0$ may or may not be essential, depending on how rapidly the sequence $\{\lambda_n\}$ grows.

Necessity here is understood in the same sense as in Theorem 1.

Let now $\{\mu_n\}$ be not necessarily a convex sequence.

Theorem 3. Let $\{\mu_n\}$ be an arbitrary strictly increasing sequence tending to infinity, and let $\{\lambda_n\}$ be some convex sequence satisfying the requirements of Theorem 2.

* The assertion of Theorem 1 concerning the case $0 < \alpha < 1/2$ and applied to trigonometric series was first announced by M. F. Timan in (8).

- 1) If $\lambda(x) \geq \mu(x)$, then the condition

$$\sum_{n=1}^{\infty} \frac{\lambda'(n)E_n}{\lambda(n)\sqrt{\lg \lambda(n)}} < \infty \quad (9)$$

is sufficient for $|R, \mu_n|$ -summability almost everywhere of orthogonal series.

- 2) If $\lambda(x) \leq \mu(x)$, $A_n(\mu) \downarrow 0$, and condition (9) is not satisfied, then the series in the Rademacher system with coefficients a_k will not be $|R, \mu_n|$ -summable almost everywhere on $[0, 1]$.
- 3) The condition $A_n(\mu) \downarrow 0$ in item 2) is essential and cannot be replaced, for example, by the condition $A_n(\lambda) \downarrow 0$.

It is interesting to note that if $\lambda(x)$ and $\mu(x)$ are arbitrary functions strictly increasing to ∞ ($\lambda(x)$ may be taken convex), $\lambda(x) \geq \mu(x)$, then the fact that series (2) is $|R, \lambda_n|$ -summable almost everywhere on $[0, 1]$ by no means implies that it will be $|R, \mu_n|$ -summable.

If $\lambda(x)$ is convex downward and $\lambda(x+1)/\lambda(x) \geq q$ for at least one $q > 1$, then the method $|R, \lambda_n^*|$ is equivalent to absolute convergence (see (1)). Consequently, for all these methods the condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} E_n < \infty \quad (10)$$

is sufficient, and for $a_n \downarrow 0$ also necessary for $|R, \lambda_n|$ -summability almost everywhere of orthogonal series.

Theorem 4. Let $\lambda(x) \geq q^x$ ($q > 1$) be an arbitrary strictly increasing function. Suppose further that condition (10) is not fulfilled and $A_n(\lambda) \downarrow 0$. Then the series in the Rademacher system is not $|R, \lambda_n^*|$ -summable almost everywhere on $[0, 1]$. The condition $A_n(\lambda) \downarrow 0$ is essential.

For $|R, \lambda_n|$ -summability one can obtain summability criteria of Weyl type.

Theorem 5. Let the sequence $\{\lambda_n\}$ satisfy the conditions of Theorem 2. If the positive function $\omega(n) \uparrow \infty$ is such that

$$\sum_{n=1}^{\infty} \frac{1}{n\omega([l'(n)])} < \infty, \quad (11)$$

then $\omega(n)$ is a Weyl multiplier of $|R, \lambda_n|$ -summability almost everywhere for orthogonal series. If inequality (11) is not satisfied, then $\omega(n)$ is not a Weyl multiplier of $|R, \lambda_n|$ -summability for trigonometric series.

The conditions (in the sufficiency part) of Theorems 2 and 5 are equivalent. From Theorem 5 it follows, in particular, that there is no exact Weyl multiplier of $|R, \lambda_n|$ -summability almost everywhere for the entire class of trigonometric series.

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