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Abstract

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PHYSICS

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ON THE INTERACTION OF WAVES IN CONTINUOUS MEDIA

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The frequencies of collective oscillations in continuous media belonging to different oscillatory branches are, as a rule, very different (examples are low-frequency and high-frequency waves in a plasma, long-wavelength acoustic and optical phonons in solids, phonons and rotons in superfluid helium, etc.). We wish to draw attention to the possibility of describing the effects of interaction of such collective oscillations by means of a self-consistent system of equations: a kinetic equation for the distribution function of high-frequency waves (in the 6-dimensional space of coordinates and wave vectors) and equations of hydrodynamic type for variations of the density, velocity, and pressure of the substance. With the aid of such a system of equations one can investigate processes whose characteristic periods and wavelengths greatly exceed the period and wavelength of the high-frequency oscillations; at the same time, only averaged characteristics of the high-frequency waves enter the equations, and under such conditions these waves may be regarded as quasiparticles.

The equations describing the interaction of quasiparticles with matter can be obtained from the Lagrangian L of a system consisting of quasiparticles and matter:

$$L = L_1 + L_2 + L_3. \tag{1}$$

Here L_1 is the Lagrangian of the matter; for example, for longitudinal sound oscillations of an isotropic medium

$$L_1 = \frac{\rho}{2}\left[\left(\frac{\partial \vec{\xi}}{\partial t}\right)^2 - c_s^2(\nabla \cdot \vec{\xi})^2\right],$$

where ρ is the density of the medium; c_s is the speed of sound; $\vec{\xi}$ is the displacement of the matter; L_2 is the Lagrangian of the quasiparticles; L_{12} is the interaction Lagrangian. For the case of interaction with longitudinal sound,

$$L_{12} = \sum_i s\,(\nabla \cdot \vec{\xi}(\mathbf{x}_i)),$$

where s is the "acoustic charge" of the quasiparticle, and the summation is over all quasiparticles. Varying (1) with respect to the quasiparticle coordinates $\mathbf{x}_i$, we obtain the force acting on a quasiparticle:

$$f = s\nabla(\nabla \cdot \vec{\xi}), \tag{2}$$

so that the Liouville equation for the quasiparticle distribution function N_k has the form

$$\frac{\partial N_k}{\partial t} + \mathbf{v}_k \cdot \nabla N_k + s\nabla(\nabla \cdot \vec{\xi})\,\frac{\partial N_k}{\partial \mathbf{k}} = 0, \tag{3}$$

where $\mathbf{v}_k = d\omega_k/d\mathbf{k}$. Further, varying (1) with respect to the field $\vec{\xi}$, we arrive at an inhomogeneous d' Alembert equation for the displacement

$$\rho\left(\frac{\partial^2 \vec{\xi}}{\partial t^2} - c_s^2 \nabla(\nabla \cdot \vec{\xi})\right) = -s\nabla \sum_{\mathbf{k}} N_k. \tag{4}$$

Equations (3) and (4) constitute the desired self-consistent system describing the interaction of high-frequency oscillations (quasiparticles) with the field of low-frequency waves.

As an example, let us consider in greater detail an isotropic plasma with hot electrons. In this case the high-frequency oscillations are Langmuir oscillations, and the low-frequency ones are ion-acoustic oscillations. In order to find the acoustic charge of the Langmuir plasmons s, we take into account that the force f acting on an individual Langmuir plasmon when the substance moves with velocity $\mathbf{u}$ is equal to

$$f = -\nabla(\omega + \mathbf{k}\mathbf{u}). \tag{5}$$

In equality (5) the term associated with the Doppler effect is much smaller than the term associated with the change in density δn, and therefore the force acting on a Langmuir plasmon is proportional to the gradient of the matter density,

$$f = -\frac{1}{2}\frac{\omega_0}{n}\nabla n, \tag{6}$$

so that $s = \frac{1}{2}\omega_0$.

The force acting on the substance from the gas of Langmuir waves can be rewritten in the form

$$-s\nabla \sum_{\mathbf{k}} N_{\mathbf{k}} = -n\nabla \frac{1}{n} \sum_{\mathbf{k}} \frac{E_{\mathbf{k}}^2}{8\pi} = -n\nabla \sum_{\mathbf{k}} \frac{e^2 E_{\mathbf{k}}^2}{2m\omega_0^2}, \tag{7}$$

where $-\nabla \dfrac{e^2 E_{\mathbf{k}}^2}{2m\omega_0^2}$ is the average force acting on an electron in the field of a Langmuir wave.

Let us consider, using the system (3) and (4), several effects.

1. **Damping of an ion-acoustic wave in a gas of Langmuir plasmons.** Let us find the damping of an ion-acoustic wave (all quantities in which vary proportionally to $e^{-i\Omega t + i\mathbf{qx}}$), caused by interaction with a gas of Langmuir plasmons. To this end, linearizing the kinetic equation (3), we determine the correction to the plasmon distribution function

$$\delta N_k = \frac{is}{\Omega - (\mathbf{q}\mathbf{v}_k)} (\mathbf{q}\xi) \left(\mathbf{q} \frac{\partial N_k}{\partial k} \right). \tag{8}$$

Substituting δN_k into equality (4), we obtain the dispersion relation connecting the frequency Ω and the wave vector $\mathbf{q}$ of the ion-acoustic wave,

$$-\Omega^2 + q^2 c_s^2 = \frac{s^2}{\rho} q^2 \int \frac{dk}{\Omega - (\mathbf{q}\mathbf{v}_k)} \left(\mathbf{q} \frac{\partial N_k}{\partial k} \right),$$

$$\frac{1}{\Omega - (\mathbf{q}\mathbf{v}_k)} = P \frac{1}{\Omega - (\mathbf{q}\mathbf{v}_k)} - i\pi\delta(\Omega - \mathbf{q}\mathbf{v}_k). \tag{9}$$

Assuming the decrement (increment) Γ_q to be small in comparison with the frequency Ω, we find from (9)

$$\Omega = \pm q c_s + i\Gamma_q, \qquad \Gamma_q = \pi \frac{s^2 q}{2\rho c_s} \int \left(\mathbf{q} \frac{\partial N_k}{\partial k} \right) \delta(\pm q c_s - \mathbf{q}\mathbf{v}_k)\, dk, \tag{10}$$

where

$$s^2 = \omega_0^2/4, \qquad \mathbf{v}_k = d\omega/dk = 3\omega_0 D^2 \mathbf{k}, \qquad D^2 = 2T_e/m\omega_0^2.$$

We note that equality (10) can be obtained by considering induced emission and absorption of ion-acoustic waves by Langmuir plasmons. The rate of change in the number of ion-acoustic waves n_q, caused by these processes, is equal to

$$\dot{n}_q = n_q \sum_{\mathbf{k}} W_{k+q/2,\,k-q/2} \left(N_{k+q/2} - N_{k-q/2} \right) \delta(\omega_{k+q/2} - \omega_{k-q/2} - \Omega_q).$$

For $q \ll k$, expanding the differences $N_{k+q/2} - N_{k-q/2}$ and $\omega_{k+q/2} - \omega_{k-q/2}$ in q/k and denoting $\dot{n}_q/n_q = 2\Gamma_q$, we arrive at equality (10), if the probability of emission-absorption is given by the formula

$$W_{k+q/2,\,k-q/2} = \frac{\pi}{4}\frac{\omega_0^2}{hMc_s}\, q. \tag{11}$$

2. Instability of a Cold Gas of Langmuir Plasmons

Let us consider small oscillations of a plasma in the presence in it of a gas of Langmuir plasmons. In analyzing the oscillations we shall use the system of hydrodynamic equations for plasmons (i.e., the system of equations relating the moments of the distribution function N_k), which can be derived from the kinetic equation (3). Integrating (3) with respect to k, we obtain the continuity equation

$$dN/dt + \nabla \cdot N\mathbf{v} = 0, \tag{12}$$

where

$$N = \int N_k \, d\mathbf{k}, \qquad \mathbf{v} = \frac{1}{N}\int \mathbf{v}_k N_k \, d\mathbf{k},$$

$\mathbf{v}$ is the mean velocity of the plasmons. Next, multiplying (3) by $\mathbf{v}_k$ and integrating with respect to k, we find the hydrodynamic equation of motion of the plasmon gas

$$\frac{d\mathbf{v}}{dt} + (\mathbf{v}\nabla)\mathbf{v} = \frac{1}{m_*}\mathbf{f} - \frac{1}{N}\nabla N \, \langle \delta \mathbf{v}^2 \rangle, \tag{13}$$

where $1/m_* = 3\omega_0 D^2$, m^* is the "effective mass" of the plasmons (D^2 is the Debye radius), and

$$\langle \delta \mathbf{v}^2 \rangle = \frac{1}{N}\int (\mathbf{v}_k - \mathbf{v})^2 N_k \, d\mathbf{k} \tag{14}$$

is the mean square spread of the group velocities in the plasmon gas. In the case of "cold" plasmons $\langle \delta \mathbf{v}^2 \rangle = 0$. Then, linearizing equations (4), (12), (13), we obtain

$$\rho(\Omega^2 - q^2 c_s^2)\,\vec{\xi} = \frac{i\omega_0}{2}\,\mathbf{q}\,\delta N,$$

$$-\Omega\,\delta N + (\mathbf{q}\cdot\mathbf{v})N = 0, \qquad -i\Omega\mathbf{v} = -\frac{1}{2m_*}\omega_0 q^2 \vec{\xi}. \tag{15}$$

From the system of equations (15) we find the dispersion relation connecting Ω and q:

$$(\Omega/q)^4 - c_s^2(\Omega/q)^2 - \beta = 0, \qquad \beta = \omega_0^2 N/4\rho m_*.$$

For $\beta \ll c_s^4$,

$$\Omega_{1,2} \simeq \pm q c_s, \qquad \Omega_{3,4} \simeq \frac{i}{\sqrt{2}}\frac{\sqrt{\beta}}{c_s}\,q \simeq \pm i\left(\sum_k E_k^2/8\pi n m\right)^{1/2} q. \tag{16}$$

It follows from equality (16) that a cold gas of plasmons is unstable with respect to the breakup of a homogeneous distribution of plasmons into separate clumps.

Let us note that if the effective mass of the quasiparticles were negative, then the solutions $\Omega_{3,4}$ would determine the phase velocity of "slow sound" in a substance in the presence of a gas of quasiparticles in it.

3.

With the aid of the dispersion relation (9) one can obtain the instability criterion for a hot gas of plasmons; the instability condition has the form

$$\langle \delta k^2 \rangle D^2 \ll \frac{N\omega_0}{nT} \simeq \sum_k E_k^2/8\pi n T, \tag{17}$$

where $\langle \delta k^2 \rangle^{1/2}$ is the spread of the values of the wave vector $\mathbf{k}$ in the plasmon gas.

4. Interaction of Plasmons with a Random Field of Ion-Sound Waves

To find the diffusion of a Langmuir plasmon in wave-number space under the action of random fields of ion-sound waves, it is necessary to take into account the nonlinear terms in the Liouville equation (3) for the distribution function N_k.

Separating the plasmon distribution function into an oscillating part (with the frequency of ion-sound oscillations) and a slowly varying "background" $\langle N_k \rangle$, and carrying out, in equation (3), an averaging over the phases of the ion-sound waves, we obtain the equation

$$\frac{d}{dt}\langle N_k\rangle = \frac{\partial}{\partial k_\alpha} D_{\alpha\beta} \frac{\partial \langle N_k\rangle}{\partial k_\beta}, \tag{18}$$

analogous to the diffusion equation in the quasilinear theory of plasma (^{1}). The diffusion coefficient $D_{\alpha\beta}$ is determined by the formula

$$D_{\alpha\beta} = \pi \sum_q s^2 q_\alpha q_\beta \, |\mathbf{q}\cdot\xi|^2 \delta(\Omega - \mathbf{q}\cdot\mathbf{v}_k). \tag{19}$$

Equation (18), describing the diffusion of plasmons in the random field of ion-acoustic waves, can be obtained independently from the kinetic equation that takes into account the process of emission–absorption of ion-acoustic waves by Langmuir plasmons. Indeed, if the occupation numbers of ion-acoustic waves n_q greatly exceed the occupation numbers of Langmuir oscillations, then the rate of change of the distribution function of Langmuir plasmons, due to the direct and inverse processes of emission and absorption of ion sound, is equal to

$$\begin{aligned}\dot{N}_k = \sum_q W_{k,k-q} n_q (N_{k-q} - N_k)\, \delta(\omega_k - \omega_{k-q} - \Omega_q) + \\ + W_{k,k+q} n_q (N_{k+q} - N_k)\, \delta(\omega_k - \omega_{k+q} + \Omega_q).\end{aligned} \tag{20}$$

Introducing the function

$$\Psi_{k,q} = W_{k,k-q}(N_k - N_{k-q})\delta(\omega_k - \omega_{k-q} - \Omega_q),$$

one can write the right-hand side of equality (20) in the form

$$\dot{N}_k = \sum_q (\Psi_{k+q,q} - \Psi_{k,q}) n_q.$$

For $q/k \ll 1$, expanding this expression in q/k, we obtain

$$\dot{N}_k = \frac{\partial}{\partial k_\alpha} D_{\alpha\beta} \frac{\partial N_k}{\partial k_\beta},$$

where the diffusion coefficient $D_{\alpha\beta}$ is related to the probability $W_{k,k-q}$ by the relation

$$D_{\alpha\beta} = \sum_q n_q q_\alpha q_\beta W_{k,k-q} \delta(\Omega - \mathbf{q}_i \cdot \mathbf{v}_k).$$

Substituting

$$W_{k,k-q} = \frac{\pi}{4}\frac{\omega_0^2}{nMc_s}\,q,$$

we arrive at equations (18)–(19). It should be noted that formulas (9) and (15), from which the instability of a gas of Langmuir plasmons follows, cannot be obtained from the kinetic equation for plasmons, which is valid only when the inequality $\Gamma \ll \Omega$ is satisfied.

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CITED LITERATURE

1. A. A. Vedenov, *Atomic Energy*, **13**, 5 (1962).

Note: Figure translations are in progress. See original paper for figures.