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**Abstract**

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# ON RIEMANN INTEGRATION IN CONSTRUCTIVE ANALYSIS

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In this note we use the definitions and results of works ([1–6]). In particular: 1) as the precise notion of an algorithm we take the notion of a normal algorithm ([1]); 2) if $\mathfrak{A}$ is some algorithm, then the symbol $\{\mathfrak{A}\}$ denotes the record of this algorithm ([6]); 3) if $P$ is the record of some algorithm, then the symbol $\langle P\rangle$ denotes this algorithm ([6]); 4) if $\mathfrak{A}$ is an algorithm in the alphabet $A$, and $P$ is a word in this alphabet, then the symbol $\mathfrak{A}_P$ denotes an algorithm, constructed in a certain way ([2]), such that $\mathfrak{A}_P(Q) \simeq \mathfrak{A}(PQ)$ for any word $Q$ in the alphabet $A$.

Below, by a "function" we shall everywhere mean a constructive function defined everywhere on the segment $0\triangle 1$. By virtue of a well-known theorem of Tseitin ([2]), every function is continuous on the segment $0\triangle 1$.

As usual, we shall call a sequence of constructive objects of a given type an algorithm that transforms each natural number into an object of this type. By a sequence of functions we shall mean a sequence of records of functions.

All our further definitions and results concern integration on the segment $0\triangle 1$; therefore mention of the segment $0\triangle 1$ will often be omitted.

1. A word $P$ of the form

$$r_0 * x_0 * r_1 * x_1 * \cdots * r_{n-1} * x_{n-1} * r_n,$$

where $r_m$ $(m = 0, \dots, n)$ are rational numbers such that $r_0 = 0$, $r_n = 1$, $r_k < r_{k+1}$, and $x_k$ $(k = 0, \dots, n-1)$ are duplexes such that $r_k \le x_k \le r_{k+1}$, will be called an **integral partition**, and

$$\max_{0\le k\le n-1}(r_{k+1} - r_k)$$

the **mesh of this integral partition**. One can construct an algorithm $\mathfrak{I}$ such that, for every integral partition $P$ and every function $f$, the equality

$$\mathfrak{I}(\{f\} \,\square\, P) = \sum_{k=0}^{n-1} f(x_k)(r_{k+1} - r_k).$$

holds.

Let $f$ be a function. Every word of the form $\{f\}\square P\square y$, where $P$ is an integral partition and $y$ is a duplex such that $y = \mathfrak{I}(\{f\}\square P)$, will be called an **integral sum** of the function $f$. The duplex $y$ and the mesh of the integral partition $P$ will be called, respectively, the **value** and the **mesh of the integral sum** $\{f\}\square P\square y$. It is easy to construct algorithms $\mathfrak{s}$ and $\pi$ that transform every integral sum, respectively, into its value and into its mesh.

We shall say that a function $f$ is **Riemann integrable** on $0\triangle 1$ if one can construct an algorithm $\delta$ of type $(\mathbb{N} \to \mathbb{N})$ such that, for any integral sums $S_1$ and $S_2$ of the function $f$ satisfying the conditions

$$\pi(S_1) < 2^{-\delta(m)} \quad \text{and} \quad \pi(S_2) < 2^{-\delta(m)},$$

the inequality

$$|\mathfrak{s}(S_1) - \mathfrak{s}(S_2)| < 2^{-m}$$

holds. In this case we shall call the algorithm $\delta$ a **regulator of integrability** of the function $f$.

A sequence $S$ of integral sums of a function $f$ will be called **convergent** if $\pi(S(n)) \to 0$ as $n \to \infty$. Having a regulator of integrability of the function $f$, one can construct a duplex $z$ such that, for any convergent sequence $S$ of integral sums of this function, $\delta(S(n)) \to z$ as $n \to \infty$. We shall call this duplex the **Riemann integral** of the function $f$.

In what follows the following lemma is essential.

**Lemma 1.** Let $\Phi$ be an exact segment, disjoint, singular covering of $0\triangle 1$ (see ([3])), $\mathfrak{D}$ an algorithm, $\omega$ a sequence of functions, $\varphi$ a function, and suppose that for every $k$ on the segments $\Phi_0, \dots, \Phi_k$ the equality

$$\varphi(x) = \langle\omega(k)\rangle(x)$$

holds. Then one can construct an algorithm $\mathfrak{A}$ such that, for every word $P$ in the alphabet of the algorithm $\mathfrak{D}$, the algorithm $\mathfrak{A}_{P\square}$ is a function and the following conditions are satisfied: if $\neg !\mathfrak{D}(P)$, then $\mathfrak{A}_{P\square}(x) = \varphi(x)$ everywhere on $0\triangle 1$; if $\mathfrak{D}$ terminates its work on $P$ in exactly $k$ steps, then $\mathfrak{A}_{P\square}(x) = \langle\omega(k)\rangle(x)$ everywhere on $0\triangle 1$.

The proof is carried out by means of the following construction. Using the theorem on a universal algorithm, we construct an algorithm $\mathfrak{B}_1$ so that

$$\mathfrak{B}_1(\xi f \supset \square x) \simeq f(x)$$

for any function $f$ and any duplex $x$. We construct an algorithm $\mathfrak{B}_2$ so that $\mathfrak{B}_2(P\square n) = k$, if $k < n$ and $\mathfrak{D}$ terminates its work on $P$ in exactly $k$ steps, and $\mathfrak{B}_2(P\square n) = \Lambda$, if $\mathfrak{D}$ does not terminate its work on $P$ in $n$ steps. By $L$ we denote the bounding algorithm of the covering $\Phi$ ([3]).

Using the theorems on the combination of algorithms, we construct an algorithm $\mathfrak{A}$ so that:

$$\mathfrak{A}(P\square x) \simeq \varphi(x), \qquad \text{if } \mathfrak{B}_2(P\square L(x)) = \Lambda;$$
$$\mathfrak{A}(P\square x) \simeq \mathfrak{B}_1(\omega(\mathfrak{B}_2(P\square L(x)))\square x), \quad \text{if } \mathfrak{B}_2(P\square L(x)) \neq \Lambda.$$

It can be shown that $\mathfrak{A}$ satisfies the conclusion of the lemma.

Taking as $\mathfrak{D}$ an algorithm with an undecidable problem of recognizing applicability, and choosing the sequences $\omega$ and the function $\varphi$ in a special way, one can prove the following theorems.

**Theorem 1.** There is no algorithm which transforms every word of the form $\xi f \supset \square u$, where $f$ is a Riemann-integrable function bounded by the number 1, and $u$ is its Riemann integral, into a record of a regulator of integrability of the function $f$.

**Theorem 2.** There is no algorithm transforming a record of every Riemann-integrable function $f$ bounded by the number 1 into the Riemann integral of the function $f$.

We shall say that a function $f$ is **effectively nonintegrable in the Riemann sense** on $0\triangle 1$, if there exist convergent sequences $S$ and $T$ of integral sums of the function $f$ and a rational number $r > 0$ such that, for every $n$, the inequality

$$|\delta(S(n)) - \delta(T(n))| \geq r$$

holds.

**Theorem 3.** Bounded functions can be constructed which are effectively nonintegrable in the Riemann sense on $0\triangle 1$.

**Remark 1.** Theorems 1–3 remain valid for the class of infinitely differentiable functions.

**Remark 2.** The functions used in the proof of Theorem 3 are effectively nonuniformly continuous on $0\triangle 1$. However, there exist effectively uniformly continuous functions on $0\triangle 1$ which are Riemann integrable. Such, for example, is the function from Theorem 5.2 ([4]).

For differentiable functions, a theorem analogous to Theorem 2 was proved by G. E. Mints ([7]). As G. S. Tseitin informed the author, Theorem 3 had been known earlier.

Using Lemma 1 and the functions constructed for the proof of Theorem 3, one can prove the following theorem.

**Theorem 4.** There is no algorithm applicable to a record of a function $f$ if and only if $f$ is Riemann integrable (nonintegrable).

(The set of Riemann-integrable (nonintegrable) functions is not recursively enumerable.)

2. In this subsection, everywhere, as the notion of integral and integrability, the general notion of integral and integrability proposed in ([3]) is used.

**Lemma 2.** *Let the function $f$ be integrable, let the duplex $u$ be its integral, and let $x$ be a point of $0\triangle 1$ such that $f(x) > u$. Then there exists a rational point $r$ of $0\triangle 1$ such that $f(r) < u$.*

It follows from this lemma that any two integrals of one and the same function are equal. Lemma 2 is used in the proof of the following theorems.

**Theorem 5.** *There is no algorithm which transforms a record of every bounded-by-1 integrable function $f$ into an integral of the function $f$.*

**Theorem 6.** *There is no algorithm which transforms every word of the form $\xi\{f\}\square u$, where $f$ is an integrable function, $u$ is its integral, into a duplex $y$ from $0\triangle 1$ such that $f(y) = u$.*

**Theorem 7.** *One can construct an algorithm which transforms every word of the form $\xi\{f\}\square u$, where $f$ is an integrable function, $u$ is its integral, into a quasinumber conditionally belonging to $0\triangle 1$ and conditionally assigning to the function $f$ a value equal to its integral.*

The proof of this theorem is analogous to the proof of Theorem 2 from § 5 of paper ([5]).

Theorem 6 shows that, in constructive analysis, the theorem on the mean value of integral calculus is not carried over in its literal formulation. However, by virtue of Theorem 7, it cannot be refuted by an example.

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*Note: Figure translations are in progress. See original paper for figures.*