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Abstract

Full Text

Mathematics

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On a System of Canonical Open (Closed) Sets

(Presented by Academician P. S. Aleksandrov on 8 I 1964)

In this note it is established which topological spaces have isomorphic systems of canonical open (closed) sets.

A set G of a topological space X is called **canonically open** if it is the open kernel of a closed set. Let $R_0(X)$ denote the system of all canonically open sets of the space X . Obviously:

$$G \in R_0(X) \iff G = \text{Int } \overline{G}.$$

By a **canonically closed** set is meant the complement of an open canonical set, or, equivalently, the closure of an open set. We denote the system of all canonically closed sets of the space X by $R_{\text{cl}}(X)$, i.e.

$$F \in R_{\text{cl}}(X) \iff F = \overline{\text{Int } F}.$$

In what follows we consider $R_0(X)$ as ordered by the inclusion relation (similarly $R_{\text{cl}}(X)$). The partially ordered sets $(R_0(X), \subset)$ and $(R_{\text{cl}}(X), \subset)$ form Boolean structures. These structures are even complete (see ⁽²⁾). For the structure operations we have:

$$(R_0(X), \subset) \quad (R_{\text{cl}}(X), \subset).$$

1. Structural union

$$G \vee H = \text{Int } \overline{(G \cup H)}, \quad F \vee K = \overline{\text{Int}(F \cup K)}.$$

2. Structural intersection

$$G \wedge H = \overline{G \cap H}, \quad F \wedge K = \overline{\text{Int}(F \cap K)}.$$

3. Structural complement

$$G' = \text{Int } CG, \quad F' = CF.$$

The operator $G \rightarrow \overline{G}$ (conversely $F \rightarrow \text{Int } F$) establishes a structural isomorphism between $(R_0(X), \subset)$ and $(R_{\text{cl}}(X), \subset)$.

As preparation for our main result, we note the following known fact about canonical open (closed) sets of an arbitrary extension.

Lemma 1. *If X is any topological space and eX is any extension of it (i.e. X is dense in eX), then X and eX possess isomorphic systems of canonical open (closed) sets. Namely:*

- a) $G \rightarrow G \cap X$ is an isomorphism between $(R_0(eX), \subset)$ and $(R_0(X), \subset)$;
- b) $H \rightarrow O(H)^*$ is the inverse isomorphism to a);
- c) $F \rightarrow F \cap X$ is an isomorphism between $(R_{\mathfrak{A}}(eX), \subset)$ and $(R_{\mathfrak{A}}(X), \subset)$;
- d) $K \rightarrow \overline{K}^{eX}$ is the inverse isomorphism to c).

Remark. P. S. Aleksandrov ⁽¹⁾ proved that the given operators map one-to-one the systems $R_0(X)$, respectively $R_{\mathfrak{A}}(X)$, of a normal space X onto the systems $R_0(\beta X)$, respectively $R_{\mathfrak{A}}(\beta X)$, of the Stone-Čech extension βX . Yu. M. Smirnov noted ⁽⁷⁾ that this is true for arbitrary X and eX . Theorem 5 of his paper ⁽⁷⁾ is equivalent to assertions a) and b) of the lemma, because the operators are, obviously, monotone.

The proof is elementary; it is based on the identities:

$$\text{For a set } O \text{ open in } eX : \quad O \cap \overline{X}^X = O^{eX} \cap X. \quad (1)$$

$$\text{For a set } F \text{ closed in } eX : \quad \text{Int}_X(F \cap X) = X \cap \text{Int}_{eX} F. \quad (2)$$

It is easy to see that formulas (1) and (2) cease to be true if one does not require in them the openness of the set O or the closedness of the set F .

* $O(H)$ is the largest open set in eX whose trace in X is H , i.e.

$$O(H) = \bigcup G \quad (G \text{ open in } eX),$$

such that $G \cap eX = H$. Obviously,

$$O(H) = eX \setminus (X \setminus H)^{eX}.$$

The operator $O(H)$ was introduced by Shanin (see ⁽⁷⁾).

The operator $G \rightarrow G \cap X$ maps $R_0(eX)$ into $R_0(X)$; namely, from relations (1) and (2) it follows that

$$\text{Int}_X(G \cap X)^X = \text{Int}_X(X \cap \overline{G}^{eX}) = X \cap \text{Int}_{eX} \overline{G}^{eX} = X \cap G.$$

On the other hand, this operator maps $R_0(eX)$ onto $R_0(X)$, because

$$H = \text{Int}_X \overline{H}^X = \text{Int}_X(X \cap \overline{H}^{eX}) = X \cap \text{Int}_{eX} (\overline{H}^{eX})$$

for $H \in R_0(X)$. For $G \in R_0(eX)$ we have

$$G = \text{Int}_{eX} \overline{(G \cap X)}^{eX};$$

hence the operator is one-to-one, and therefore (since the operator is obviously monotone) $G \rightarrow G \cap X$ gives an isomorphism between $(R_0(eX), \subset)$ and $(R_0(X), \subset)$.

For the other operators we note only the relations

$$O(H) = \text{Int}_{eX}(\overline{H}^{eX})$$

for $H \in R_0(X)$, and

$$\overline{F \cap X}^{eX} = \overline{F}$$

for $F \in R_{\mathfrak{A}}(eX)$.

Lemma 2*. Let $f : X \rightarrow Y$ be a mapping of the space X onto the space Y . If: 1) f is closed, 2) irreducible, **3) for every neighborhood V of any point $f(x)$ there exists a neighborhood U of the point x such that $f(U) \subset V^*$,** then the operator $F \rightarrow f(F)$ is a structural isomorphism between $(R_{\mathfrak{A}}(X), \subset)$ and $(R_{\mathfrak{A}}(Y), \subset)$.

The proof consists of three steps:

$$\text{For every } F \in R_{\mathfrak{A}}(X) \text{ we have } f(F) = C \text{Int } f(F'). \quad (1)$$

$$\text{For every } H \in R_{\mathfrak{A}}(Y) \text{ we have } f^{-1}(\text{Int } H) \in R_{\mathfrak{A}}(X), \quad f(f^{-1}(\text{Int } H)) = H. \quad (2)$$

$$\text{For } F_1, F_2 \in R_{\mathfrak{A}}(X) \text{ we have } f(F_1) = f(F_2) \text{ when } F_1 \neq F_2. \quad (3)$$

Step 1. Since $F \in R_{\mathfrak{A}}(X)$, $CF = \text{Int } F'$; consequently, from

$$Cf(F) \subset f(CF)$$

it follows that

$$Cf(F) \subseteq f(F');$$

f is closed, and therefore $Cf(F)$ is open, and

$$Cf(F) \subseteq \text{Int } f(F'),$$

i.e.

$$f(F) \supseteq C \text{Int } f(F').$$

Now let $y \in f(F)$, and suppose that $y \in \text{Int } f(F')$. For some $x \in F$ we have $f(x) = y$. From (3) there follows the existence of an open neighborhood V of the point x such that

$$f(V) \subseteq \overline{\text{Int } f(F')} \subseteq f(F').$$

The open set V intersects $\text{Int } F$. From (2) there follows the existence of a point $y_1 \in Y$ for which

$$f^{-1}(y_1) \subseteq V \cap \text{Int } F.$$

But the point y_1 must be in $f(F')$, although

$$f^{-1}(y_1) \cap F = \emptyset.$$

Step 2. Let $H \in R_{\mathfrak{A}}(Y)$ and $x \in f^{-1}(\text{Int } H)$. From (3) there follows the existence of an open neighborhood U of the point x such that

$$f(U) \subseteq \overline{\text{Int } H} = H.$$

Then

$$U \subseteq f^{-1}(\text{Int } H),$$

because otherwise the open set

$$V = U \cap C\overline{f^{-1}(\text{Int } H)}$$

would contain $f^{-1}(zy)$ for some $y \in Y$. Then

$$y \in f(\overline{f^{-1}(\text{Int } H)}) \supset \text{Int } H.$$

Further,

$$f(\overline{f^{-1}(\text{Int } H)})$$

is closed; consequently

$$f(\overline{f^{-1}(\text{Int } H)}) \supseteq \overline{\text{Int } H} = H,$$

and $y \in H$, although $y \in f(U) \subseteq H$. Hence

$$f^{-1}(\text{Int } H) \subseteq \text{Int } \overline{f^{-1}(\text{Int } H)},$$

therefore

$$f^{-1}(\text{Int } H) \in R_{\mathfrak{A}}(X).$$

To prove the equality

$$f(\overline{f^{-1}(\text{Int } H)}) = H$$

it remains only to prove

$$H \supseteq f(\overline{f^{-1}(\text{Int } H)}).$$

Suppose, for some

$$x \in \overline{f^{-1}(\text{Int } H)},$$

that $f(x) \in H$. Then there exists a neighborhood U of the point x such that

$$f(U) \subseteq CH,$$

i.e.

$$f(U) \cap \text{Int } H = \emptyset.$$

But

$$U \cap f^{-1}(\text{Int } H) \neq \emptyset.$$

Step 3. If $F_1, F_2 \in R_{\mathfrak{A}}(X)$ and $F_1 \neq F_2$, then

$$\text{Int } F_1 \cap CF_2 \neq \emptyset$$

or

$$\text{Int } F_2 \cap CF_1 \neq \emptyset;$$

suppose

$$\text{Int } F_1 \cap CF_2 \neq \emptyset.$$

Then there exists a point $y \in Y$ for which

$$f^{-1}(y) \subseteq \text{Int } F_1 \cap CF_2,$$

therefore $y \in f(F_1)$, but $y \in f(F_2)$.

Relations (1) and (3) give

$$f(R_{\mathfrak{A}}(X)) = R_{\mathfrak{A}}(Y),$$

and (3) gives the one-to-one character of the operator $F \rightarrow f(F)$. Moreover, this operator is monotone, which completes the proof.

* See V. I. Ponomarev (8), pp. 106-107, propositions A-B, containing Lemma 2 in the assertion of a continuous (irreducibly closed) mapping f .

** f is called irreducible if, for every closed F in X , $F \neq X$, it is true that $f(F) \neq Y$.

*** For a regular space Y , condition 3) is equivalent to the continuity of f . If in 3) the existence of a closed neighborhood U is required, then 3) is equivalent to the θ -continuity defined by Fomin (see (4)). For a regular source space X , 3) is equivalent to the θ -continuity of f .

Using this, we shall now give a purely topological characterization of the class of Hausdorff spaces that possess isomorphic Boolean structures of their canonical open sets.

Theorem. Let B be a complete Boolean structure, and let $K(B)$ be the class of all Hausdorff spaces X for which the Boolean structures $R_0(X)$ of canonical open sets are isomorphic to B .

A Hausdorff space X belongs to $K(B)$ if and only if it is the image of some dense subspace of the representation space $M(B)^*$ of the structure B under a closed, irreducible, and θ -continuous mapping.

Proof. Let $X \in K(B)$. In paper (3) we proved that the space $\overset{0}{X}$, belonging to the space X , of all fixed regular ultrafilters is dense in $M(B)$, and that the canonical mapping

$$\varphi : \overset{0}{X} \rightarrow X$$

is θ -continuous, irreducible, and closed. If, conversely, the space X is the θ -continuous, irreducible, closed image of a dense subspace D of the space $M(B)$, then it follows from Lemma 2 that

$$R_{\mathfrak{A}}(X) \cong R_{\mathfrak{A}}(D),$$

and further, by Lemma 1,

$$R_{\mathfrak{A}}(D) \cong R_{\mathfrak{A}}(M(B)).$$

From the completeness of the structure B it follows that $M(B)$ is extremally disconnected; consequently every canonical closed set in $M(B)$ is even open-closed. Therefore

$$R_{\mathfrak{A}}(M(B)) \cong B$$

Remark. In this connection we refer to Birkhoff's problem 82 (2): to find necessary and sufficient conditions for a given Boolean structure B to be isomorphic to the structure $R_0(X)$ of a suitable T_1 -space or a suitable metric space.

The first part of Birkhoff's problem is solved quite simply by establishing that complete Boolean structures, and only they, can be represented in the form $R_0(X)$, where X is a T_1 -space. Namely, their representation space gives such a realization (this was already noted in the proof of the theorem).

The next problem—to find necessary and sufficient conditions for a complete Boolean structure to be representable in the form $R_0(X)$ for a space X of a given type—receives, in view of our theorem, the following topological analogue:

For compact extremally disconnected T_2 -spaces X , find conditions under which at least one representative of the given type is contained in the class of all θ -continuous, irreducible, closed images of its dense subspaces.

In Birkhoff's original problem, a type of metric spaces is required.

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* The Stone structure space $M(B)$, or the representation space of the structure B , is the unique zero-dimensional ($\text{ind} = 0$) Hausdorff compact space for which the (elementary) structure of open-closed sets is isomorphic to B . It is constructed as the space of ultrafilters (or maximal ideals) of the structure B .

** With the aid of the space $\overset{0}{X}$ belonging to the space X , we may say: for Hausdorff spaces X and Y ,

$$R_0(X) \cong R_0(Y) \iff \beta\overset{0}{X} \cong \beta\overset{0}{Y}$$

(because

$$M(R_0(X)) \cong \beta\overset{0}{X}.$$

Here it is essential that $\overset{0}{X}$ admits a purely topological description (see Lemma 2 and Theorem 1 of paper ⁽³⁾).

Note: Figure translations are in progress. See original paper for figures.

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