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Abstract

Full Text

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CORRELATION FUNCTIONS OF A GAUSSIAN MARKOV PROCESS

(Presented by Academician S. N. Bernstein on January 3, 1964)

1. In this article a two-dimensional stationary Gaussian Markov process $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$ is considered. It is assumed that $M\xi_1(t) = M\xi_2(t) = 0$, $M\xi_1^2(t) = M\xi_2^2(t) = 1$. By $R(t) = \|r_{ij}(t)\|$ we denote the correlation matrix function, where $r_{ij}(t)$ is defined as follows:

$$r_{ij}(t) = M\xi_i(s)\xi_j(s+t), \quad i, j = 1, 2.$$

The matrix function $R(t)$ satisfies the equation $r_{ij}(t) = r_{ji}(-t)$. Therefore it is sufficient to investigate $R(t)$ for $t \geq 0$. Below, t is everywhere considered nonnegative. The process is assumed to be nondegenerate ($r = r_{12}(0) = r_{21}(0) \neq \pm 1$) and nonsingular. The correlation functions $r_{ij}(t)$ must satisfy the conditions:

- I. $|r_{ij}(t)| \leq 1$, since the process is normalized.

- II. The principal minors of the matrix

$$S(t) = \begin{vmatrix} 1 & r & r_{11}(t) & r_{12}(t) \\ r & 1 & r_{21}(t) & r_{22}(t) \\ r_{11}(t) & r_{21}(t) & 1 & r \\ r_{12}(t) & r_{22}(t) & r & 1 \end{vmatrix}$$

must be positive.

The purpose of this article is to determine which matrix functions can be correlation matrix functions of a Gaussian Markov process. In doing so, we shall start from conditions I and II and from the system of functional equations for the correlation functions $r_{ij}(t)$ of a Gaussian Markov process*:

$$(1 - r^2)r_{11}(t_1 + t_2) = r_{11}(t_1)r_{11}(t_2) + r_{12}(t_1)r_{21}(t_2) - r[r_{12}(t_1)r_{11}(t_2) + r_{11}(t_1)r_{21}(t_2)]; \quad (1)$$

$$(1 - r^2)r_{22}(t_1 + t_2) = r_{22}(t_1)r_{22}(t_2) + r_{21}(t_1)r_{12}(t_2) - r[r_{21}(t_1)r_{22}(t_2) + r_{22}(t_1)r_{12}(t_2)]; \quad (2)$$

$$(1 - r^2)r_{12}(t_1 + t_2) = r_{12}(t_1)r_{22}(t_2) + r_{11}(t_1)r_{12}(t_2) - r[r_{12}(t_1)r_{12}(t_2) + r_{11}(t_1)r_{22}(t_2)]; \quad (3)$$

$$(1 - r^2)r_{21}(t_1 + t_2) = r_{21}(t_1)r_{11}(t_2) + r_{22}(t_1)r_{21}(t_2) - r[r_{21}(t_1)r_{21}(t_2) + r_{22}(t_1)r_{11}(t_2)]. \quad (4)$$

2. From the system of functional equations (1)–(4) one can pass to a system of differential equations with constant coefficients, which agrees with the remark made in work ⁽¹⁾:

$$(1 - r^2)r'_{11}(t) = [r'_{11}(0) - rr'_{21}(0)]r_{11}(t) + [r'_{21}(0) - rr'_{11}(0)]r_{12}(t); \quad (5)$$

$$(1 - r^2)r'_{12}(t) = [r'_{12}(0) - rr'_{22}(0)]r_{11}(t) + [r'_{22}(0) - rr'_{12}(0)]r_{12}(t); \quad (6)$$

$$(1 - r^2)r'_{21}(t) = [r'_{11}(0) - rr'_{21}(0)]r_{21}(t) + [r'_{21}(0) - rr'_{11}(0)]r_{22}(t); \quad (7)$$

$$(1 - r^2)r'_{22}(t) = [r'_{12}(0) - rr'_{22}(0)]r_{21}(t) + [r'_{22}(0) - rr'_{12}(0)]r_{22}(t). \quad (8)$$

The system of equations (5)–(8) splits into two systems of equations: the system (5)–(6) and the system (7)–(8). Both of these systems of equations have one and the same characteristic equation

$$\begin{vmatrix} r'_{11}(0) - rr'_{21}(0) - k & r'_{21}(0) - rr'_{11}(0) \\ r'_{12}(0) - rr'_{22}(0) & r'_{22}(0) - rr'_{12}(0) - k \end{vmatrix} = 0. \quad (9)$$

* The equations were communicated to me by O. V. Sarmanov.

If equation (9) has two real distinct roots $k_1 = \lambda$, $k_2 = \mu$, then the solution of system (5)–(8) consists of linear combinations of the two exponential functions $e^{\lambda t}$ and $e^{\mu t}$. If the root of equation (9) is multiple, $k_{1,2} = \lambda$, then the solution of system (5)–(8) consists of linear combinations of the two functions $e^{\lambda t}$ and $te^{\lambda t}$. If, however, the roots of (9) are complex: $k_{1,2} = \lambda \pm i\nu$, then the solution of system (5)–(8) consists of linear combinations of the functions $e^{\lambda t} \cos \nu t$ and $e^{\lambda t} \sin \nu t$. Since $r_{ij}(t)$ are bounded, $\operatorname{Re} k_{1,2} \leq 0$.

Below it will be clarified under what additional restrictions the solution of system (9)–(12) satisfies conditions I and II.

3. Consider the case when $r_{ij}(t)$ have the form

$$r_{ij}(t) = a_{ij}e^{\lambda t} + b_{ij}e^{\mu t}, \quad (10)$$

where $a_{ij} + b_{ij} = r_{ij}(0)$.

Lemma 1. The coefficients a_{ij} satisfy the equations

$$(1 - r^2)a_{11} = a_{11}^2 + a_{12}a_{21} - ra_{11}(a_{12} + a_{21}); \quad (11)$$

$$(1 - r^2)a_{22} = a_{22}^2 + a_{12}a_{21} - ra_{22}(a_{12} + a_{21}); \quad (12)$$

$$(1 - r^2)a_{12} = a_{12}(a_{11} + a_{22}) - r(a_{11}a_{22} + a_{12}^2); \quad (13)$$

$$(1 - r^2)a_{21} = a_{21}(a_{11} + a_{22}) - r(a_{11}a_{22} + a_{21}^2). \quad (14)$$

Lemma 2. If at least one of the inequalities $a_{11} \neq a_{22}$, $a_{12} \neq a_{21}$ holds, then the relations

$$1 - r^2 = a_{11} + a_{22} - r(a_{12} + a_{21}), \quad a_{11}a_{22} = a_{12}a_{21}. \quad (15)$$

hold.

Lemma 3. If $a_{11} = a_{22}$ and $a_{12} = a_{21}$, then either $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = r$, or $b_{11} = b_{22} = 1$, $b_{12} = b_{21} = r$.

Theorem 1. Let $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$ be a Gaussian Markov process with correlation functions (10). Then, by a nondegenerate linear transformation

$$\tilde{\xi}_1(t) = \alpha\xi_1(t) + \beta\xi_2(t), \quad \tilde{\xi}_2(t) = \gamma\xi_1(t) + \delta\xi_2(t) \quad (16)$$

of the random variables $\xi_1(t), \xi_2(t)$, one can obtain a normalized Gaussian Markov process $\{\tilde{\xi}(t)\} = \{(\tilde{\xi}_1(t), \tilde{\xi}_2(t))\}$, whose correlation functions have the form

$$\tilde{r}_{11}(t) = e^{\lambda t}, \quad \tilde{r}_{12}(t) = \tilde{r}e^{\lambda t}, \quad \tilde{r}_{21}(t) = \tilde{r}e^{\mu t}, \quad \tilde{r}_{22}(t) = e^{\mu t}; \quad (17)$$

the correlation coefficient \tilde{r} is determined by the formula

$$\tilde{r} = \alpha\gamma + r(\alpha\delta + \beta\gamma) + \beta\delta. \quad (18)$$

The transformation matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is determined from the system of equations

$$\alpha^2 b_{11} + \alpha\beta(b_{12} + b_{21}) + \beta^2 b_{22} = 0, \quad (19)$$

$$\alpha\gamma b_{11} + \alpha\delta b_{12} + \beta\gamma b_{21} + \beta\delta b_{22} = 0, \quad (20)$$

$$\alpha\gamma a_{11} + \alpha\delta a_{21} + \beta\gamma a_{12} + \beta\delta a_{22} = 0, \quad (21)$$

$$\gamma^2 a_{11} + \gamma\delta(a_{12} + a_{21}) + \delta^2 a_{22} = 0, \quad (22)$$

$$\alpha^2 + 2r\alpha\beta + \beta^2 = 1, \quad \gamma^2 + 2r\gamma\delta + \delta^2 = 1. \quad (23)$$

Relying on Lemma 2, one can prove that a solution of system (19)–(23) exists and is uniquely determined.

Remark. If $\lambda = \mu$, then the process $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$ is a process whose correlation functions already have the form (17).

Theorem 1 makes it possible to reduce the investigation of correlation functions of the form (10) to the investigation of correlation functions $\tilde{r}_{ij}(t)$ of the form (17). Let a Gaussian Markov process have functions (17) as its correlation functions. It turns out that condition II holds if and only if the inequality

$$\frac{\tilde{r}^2}{(1 + \sqrt{1 - \tilde{r}^2})^2} \leq \frac{\lambda}{\mu} \leq \frac{(1 - \sqrt{1 - \tilde{r}^2})^2}{\tilde{r}^2} \quad (24)$$

holds.

Therefore the following holds.

Theorem 2. Let $R(t) = \|r_{ij}(t)\|$ be a matrix function, where $r_{ij}(t)$ have the form (10), with $|r| < 1$. Then $R(t)$ is the correlation matrix function of a Gaussian Markov process if and only if: 1) the coefficients a_{ij} satisfy equations (11)–(14); 2) λ and μ satisfy inequality (24), where \tilde{r} is defined according to (18), and the coefficients $\alpha, \beta, \gamma, \delta$ are determined from the system (19)–(23).

Remark. If $\tilde{r} = 0$, then inequality (24) imposes no restrictions on λ and μ .

4. Let the correlation functions $r_{ij}(t)$ have the form:

$$r_{ij}(t) = e^{\lambda t} [r_{ij}(0) + a_{ij}t]. \quad (25)$$

The case in which all a_{ij} vanish was considered in Sec. 3. In Sec. 4 we shall consider the case in which at least one of the coefficients a_{ij} is nonzero.

Lemma 4. The coefficients a_{ij} satisfy the equations

$$a_{11}^2 + a_{12}a_{21} - ra_{11}(a_{12} + a_{21}) = 0; \quad (26)$$

$$a_{22}^2 + a_{12}a_{21} - ra_{22}(a_{12} + a_{21}) = 0; \quad (27)$$

$$r(a_{11}a_{22} + a_{12}^2) - a_{12}(a_{11} + a_{22}) = 0; \quad (28)$$

$$r(a_{11}a_{22} + a_{21}^2) - a_{21}(a_{11} + a_{22}) = 0. \quad (29)$$

Lemma 5. The coefficients a_{12} and a_{21} are not equal to each other.

Lemma 6. The coefficients a_{ij} satisfy the equations

$$a_{11} + a_{22} = r(a_{12} + a_{21}), \quad a_{11}a_{22} = a_{12}a_{21}. \quad (30)$$

Theorem 3. Let $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$ be a Gaussian Markov process with correlation functions (25). Then there exists a linear transformation (16) of the process $\{\xi(t)\}$ such that the process $\{\tilde{\xi}(t)\} = \{(\tilde{\xi}_1(t), \tilde{\xi}_2(t))\}$ is a Gaussian Markov process with correlation functions

$$\tilde{r}_{11}(t) = \tilde{r}_{22}(t) = e^{\lambda t}, \quad \tilde{r}_{21}(t) = 0, \quad \tilde{r}_{12}(t) = \tilde{a}_{12}te^{\lambda t}. \quad (31)$$

The coefficient \tilde{a}_{12} is expressed by the formula

$$\tilde{a}_{12} = \alpha\gamma a_{11} + \alpha\delta a_{12} + \beta\gamma a_{21} + \beta\delta a_{22}. \quad (32)$$

Let us note that if $a_{11} = a_{22}$, then, according to Lemma 6, $a_{12}a_{21} = 0$. Without loss of generality one may assume that $a_{21} = 0$. Then from equation (28) it follows that $ra_{12}^2 = 0$. Since not all a_{ij} vanish in the case under consideration, $r = 0$. The process $\{\xi(t)\}$ is the desired one. In what follows, without loss of generality, one may assume that $a_{11} \neq 0$. Transformation (16) is determined as follows:

$$\alpha = -\frac{a_{12}}{a_{11}}\beta; \quad \gamma = -\frac{a_{21}}{a_{11}}\delta; \quad \beta = \frac{a_{11}}{\sqrt{a_{12}^2 - 2ra_{11}a_{12} + a_{11}^2}};$$

$$\delta = \frac{a_{11}}{\sqrt{a_{21}^2 - 2ra_{11}a_{21} + a_{11}^2}}. \quad (33)$$

Let $R(t) = \|r_{ij}(t)\|$, where $r_{ij}(t)$ have the form (25), be the correlation matrix function of the process $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$. Then $\tilde{R}(t) = \|\tilde{r}_{ij}(t)\|$, where $\tilde{r}_{ij}(t)$ is defined according to (31), is the correlation matrix function of the Gaussian process $\{\tilde{\xi}(t)\} = \{(\tilde{\xi}_1(t), \tilde{\xi}_2(t))\}$. The process $\{\tilde{\xi}(t)\}$ is centered and normalized. The function $\tilde{R}(t)$ must satisfy conditions I and II of Sec. 3. However, $\tilde{R}(t)$ satisfies conditions I and II if and only if

$$|\tilde{a}_{12}| \leq -2\lambda. \quad (34)$$

Theorem 4. Let $R(t) = \|r_{ij}(t)\|$ be a matrix function, where $r_{ij}(t)$ have the form (25). The matrix function $R(t)$ is a correlation

function of a Gaussian Markov process if and only if (34) holds and the coefficients a_{ij} satisfy (26)–(29).

5. Let the correlation functions $r_{ij}(t)$ have the form:

$$r_{ij}(t) = e^{\lambda t} [r_{ij}(0) \cos \nu t + a_{ij} \sin \nu t], \quad i, j = 1, 2. \quad (35)$$

Lemma 7. The coefficients a_{ij} satisfy the equations

$$1 - r^2 = -a_{11}^2 - a_{12}a_{21} + ra_{11}(a_{12} + a_{21}); \quad (36)$$

$$1 - r^2 = -a_{22}^2 - a_{12}a_{21} + ra_{22}(a_{12} + a_{21}); \quad (37)$$

$$r(1 - r^2) = ra_{11}a_{22} + ra_{12}^2 - a_{12}(a_{11} + a_{22}); \quad (38)$$

$$r(1 - r^2) = ra_{11}a_{22} + ra_{21}^2 - a_{21}(a_{11} + a_{22}). \quad (39)$$

Lemma 8. The coefficients a_{ij} satisfy the equation

$$a_{11} + a_{22} = r(a_{12} + a_{21}). \quad (40)$$

Theorem 5. Let $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$ be a Gaussian Markov process with correlation functions (35). Then by a linear nonsingular transformation (16) of the random variables $\xi_1(t), \xi_2(t)$ one can obtain a Gaussian Markov process $\{\tilde{\xi}(t)\} = \{(\tilde{\xi}_1(t), \tilde{\xi}_2(t))\}$ whose correlation functions have the form

$$\tilde{r}_{11}(t) = \tilde{r}_{22}(t) = e^{\lambda t} \cos \nu t, \quad \tilde{r}_{12}(t) = \tilde{a}_{12} e^{\lambda t} \sin \nu t, \quad (41)$$

$$\tilde{r}_{21}(t) = \tilde{a}_{21} e^{\lambda t} \sin \nu t.$$

In this case

$$\tilde{a}_{12} = \alpha\gamma a_{11} + \alpha\delta a_{12} + \beta\gamma a_{21} + \beta\delta a_{22}; \quad (42)$$

$$\tilde{a}_{21} = \alpha\gamma a_{11} + \beta\delta a_{21} + \alpha\delta a_{21} + \beta\delta a_{22}. \quad (43)$$

The matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of the nonsingular transformation (16) is determined by equations (23) and the equations

$$a_{11}\alpha^2 + (a_{12} + a_{21})\alpha\beta + a_{22}\beta^2 = 0, \quad (44)$$

$$a_{11}\gamma^2 + (a_{12} + a_{21})\gamma\delta + a_{22}\delta^2 = 0, \quad (45)$$

up to interchange of the rows of the matrix A .

Theorem 5 makes it possible to reduce the study of the correlation functions of the Gaussian Markov process $\{\xi(t)\} = \{(\xi_1(t), \xi_2(t))\}$, having the form (35), to the study of the correlation functions $\tilde{r}_{ij}(t)$ of the corresponding Gaussian Markov process $\{\tilde{\xi}(t)\} = \{(\tilde{\xi}_1(t), \tilde{\xi}_2(t))\}$, having the form (41). The correlation matrix function $\tilde{R}(t) = \|\tilde{r}_{ij}(t)\|$ must satisfy conditions I and II. It turns out that condition I is fulfilled if and only if the inequalities

$$\min\{|\tilde{a}_{12}|, |\tilde{a}_{21}|\} \geq \frac{\nu}{\sqrt{\lambda^2 + \nu^2}} \exp\left\{\frac{\lambda}{\nu} \operatorname{arctg}\left(-\frac{\nu}{\lambda}\right)\right\}; \quad (46)$$

$$\max\{|\tilde{a}_{12}|, |\tilde{a}_{21}|\} \leq \frac{\sqrt{\lambda^2 + \nu^2}}{\nu} \exp\left\{-\frac{\lambda}{\nu} \operatorname{arctg}\left(-\frac{\nu}{\lambda}\right)\right\}. \quad (47)$$

As for condition II, it is fulfilled if and only if

$$\nu^2(\tilde{a}_{12} + \tilde{a}_{21})^2 \leq 4\lambda^2. \quad (48)$$

Thus the following holds.

Theorem 6. Let the functions $r_{ij}(t)$ have the form (35). The functions $r_{ij}(t)$ form the correlation matrix function of a normalized and centered Gaussian Markov process if and only if: 1) the coefficients a_{ij} satisfy equations (36)–(39); 2) inequalities (46)–(48) hold, where \tilde{a}_{12} and \tilde{a}_{21} are determined according to (42) and (43). The transformation matrix (16) is determined from equations (23), (44), and (45).

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Note: Figure translations are in progress. See original paper for figures.

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