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Abstract

Full Text

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ON THE SINGULARITIES OF TAYLOR-DIRICHLET SERIES

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We shall assume that the series

$$f(z) = \sum_{n=1}^{\infty} a_n z^{m_n} e^{-\lambda_n z}, \tag{1}$$

where m_n are natural numbers or zeros, $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$, has a nonempty domain of convergence. Let τ be the upper density of the sequence $\{\lambda_n\}$, taking into account that the term λ_n has multiplicity $m_n + 1$, i.e.

$$\tau = \overline{\lim_{n\to\infty}} \frac{\sum_{k=1}^{n}(m_k+1)}{\lambda_n},$$

with $\tau < \infty$.

Construct an entire function of exponential type $\varphi(z)$, satisfying the conditions

$$\varphi(\lambda_n) = \varphi'(\lambda_n) = \cdots = \varphi^{(m_n-1)}(\lambda_n) = 0, \tag{2}$$

$$\varphi^{(m_n)}(\lambda_n) = (-1)^{m_n}.$$

From the theorem proved by G. P. Lapin ([1]), and from equality (2), it follows that the function $\varphi(z)$ satisfying these conditions can be found among functions of order not higher than the first and of type $\pi\tau + \gamma^+$, where $\gamma^+ = \max(\gamma, 0)$,

$$\gamma = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \ln\left|\frac{\gamma_n}{m_n!}\right|, \qquad \gamma_n = \left[\frac{(z-\lambda_n)^{m_n+1}}{L(z)}\right]_{z=\lambda_n},$$

$$L(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)^{m_n+1}.$$

Since

$$\frac{L(z)}{(z-\lambda_n)^{m_n+1}} = \frac{(z+\lambda_n)^{m_n+1}L(z)}{(-1)^{m_n+1}\lambda_n^{2(m_n+1)}(1-z^2/\lambda_n^2)^{m_n+1}},$$

we have

$$\left[\frac{L(z)}{(z-\lambda_n)^{m_n+1}}\right]_{z=\lambda_n} = (-1)^{m_n+1}\frac{2^{m_n+1}}{\lambda_n^{m_n+1}}L_n(\lambda_n),$$

where

$$L_n(z) = \prod_{k=1}^{n-1}\left(1-\frac{z^2}{\lambda_k^2}\right)^{m_k+1} \cdot \prod_{k=n+1}^{\infty}\left(1-\frac{z^2}{\lambda_k^2}\right)^{m_k+1}.$$

It is easy to see that

$$L^{(m_n+1)}(\lambda_n) = (-1)^{m_n+1}\frac{(m_n+1)!\,2^{m_n+1}}{\lambda_n^{m_n+1}}L_n(\lambda_n),$$

and therefore

$$\frac{1}{\gamma_n} = \left[\frac{L(z)}{(z-\lambda_n)^{m_n+1}}\right]_{z=\lambda_n} = \frac{1}{(m_n+1)!}\,L^{(m_n+1)}(\lambda_n).$$

Thus,

$$\left|\frac{\gamma_n}{m_n!}\right| = \left|\frac{m_n+1}{L^{(m_n+1)}(\lambda_n)}\right|,$$

and since

$$\lim_{n\to\infty}\frac{\ln(m_n+1)}{\lambda_n} = 0,$$

we have

$$\gamma = \overline{\lim_{n\to\infty}}\,\frac{1}{\lambda_n}\ln\left|\frac{1}{L^{(m_n+1)}(\lambda_n)}\right|.$$

We shall call the number γ the **condensation index** of the sequence $\{\lambda_n\}$, taking into account the multiplicity of its terms.

It was proved by the author [2] that if G is the domain of holomorphy of the function $f(z)$, and $\varphi(z)$ is an entire function of order not exceeding one and of type σ, then the function

$$F(z) = \sum_{n=1}^{\infty} a_n \left[z^{m_n}\varphi(\lambda_n) - \binom{m_n}{1} z^{m_n-1}\varphi'(\lambda_n) + \cdots \right.$$

$$\left. \cdots + (-1)^{m_n}\varphi^{(m_n)}(\lambda_n) \right] e^{-\lambda_n z}$$

is holomorphic in the domain G_σ, which we obtain if we remove from the domain G all disks of radius σ with centers at boundary points of the domain G (G_σ may consist of several connected components). In the case under consideration $\sigma \le \pi + \gamma^+$, and, by virtue of (2),

$$F(z) = \sum_{n=1}^{\infty} d_n e^{-\lambda_n z}. \tag{3}$$

If D is the upper density, and δ is the condensation index of the sequence $\{\lambda_n\}$ without taking into account the multiplicity of its terms, i.e.

$$D = \overline{\lim_{n\to\infty}} \frac{n}{\lambda_n}, \qquad \delta = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \ln \left| \frac{1}{\overline{L}'(\lambda_n)} \right|,$$

where

$$\overline{L}(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

and $\delta^+ = \max(\delta, 0)$, then, as is known [3], in the closed disk of radius $\pi D + \delta^+$ with center at any point of the axis of convergence (the boundary of the half-plane of convergence) of the Dirichlet series (3) there is at least one singular point of the function $F(z)$. Thus we have proved the following assertion.

Theorem. *In every closed disk of radius $\pi(D + \tau) + \delta^+ + \gamma^+$ with center at a point lying on the axis of convergence of the series (3), there is at least one singular point of the function $f(z)$, defined by the series (1).*

We note that, since $\lim\limits_{n\to\infty} \dfrac{\ln n}{\lambda_n} = 0$, the abscissa of convergence k of series (3) is computed by the formula

$$k = \overline{\lim_{n\to\infty}} \frac{\ln |a_n|}{\lambda_n}.$$

If the limit $\lim\limits_{n\to\infty}\dfrac{n}{\lambda_n}$ exists, then from the known theorems for Dirichlet series (4) it follows that the axis of holomorphy (the boundary of the half-plane of holomorphy) of series (3) is at a distance not greater than δ from the axis of convergence (in this case $\delta^+ = \delta$), and on every segment of the axis of holomorphy of length $2\pi D$ there is at least one singular point of the function $F(z)$. Therefore, in this case the theorem proved can be sharpened:

On every segment of the axis of convergence of series (3) of length $2\pi D$ there is a point such that in the circle of radius $\pi\tau + \gamma^+ + \delta$ centered at this point there is at least one singular point of the function $f(z)$.

In the case when the limit

$$\tau = \lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}(m_k+1)}{\lambda_n}$$

exists (then $\lim\limits_{n\to\infty}\dfrac{m_n}{\lambda_n} = 0$, and series (1), (3) have as their domain of convergence one and the same half-plane), it follows from what was proved by A. F. Leont' ev (5) that the aforementioned classical theorems for Dirichlet series extend to series (1), and in their formulations the density and condensation index of the sequence $\{\lambda_n\}$ should be taken with account of the multiplicities of the terms of this sequence.

Finally, we note that the theorem proved will remain valid also when the λ_n are complex numbers (let, for simplicity, $\lambda_n^2 \neq \lambda_k^2$ for $n \neq k$), if in the definitions of the numbers τ, D, and in the denominators standing under the limit sign of the quantities in the definitions of the numbers δ and γ, λ_n is replaced by $|\lambda_n|$.

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Note: Figure translations are in progress. See original paper for figures.