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**Abstract**

**Full Text**

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## **SMOOTH STRUCTURES ON MANIFOLDS WITH BOUNDARY**

*(Presented by Academician P. S. Aleksandrov, 14 II 1964)*

Let  $W^n$ ,  $n > 5$ ,  $n \neq 6, 14$ , denote an  $n$ -dimensional simply connected smooth manifold with boundary  $\partial W^n = V^{n-1}$ ; the boundary is assumed to be simply connected. The aim of this paper is to describe the class of manifolds homotopy equivalent to the given manifold  $W^n$  and with equivalent normal bundles.

More precisely, the following formulations of the problem are possible:

**P1.** Describe, up to diffeomorphism, the set of all smooth manifolds  $\{W'^n\}_1$  with boundary  $\partial W'^n = V'^{n-1}$  such that: a) for a given  $W'^n$  there exists a map of pairs

$$F : (W'^n, \partial W'^n) \rightarrow (W^n, \partial W^n),$$

which is a homotopy equivalence; b) the map  $F$ , considered on the boundary  $\partial W'^n$ , is a diffeomorphism of degree  $+1$

$$F|_{\partial W'^n} : \partial W'^n \rightarrow \partial W^n;$$

c) if  $\nu_W$  is the stable normal bundle of the manifold  $W^n$  and  $\nu_{W'}$  is the stable normal bundle of the manifold  $W'^n$ , then  $F^*\nu_W = \nu_{W'}$ .

**P2.** A second formulation of the problem is obtained by considering the set  $\{W'^n\}_2$  and retaining conditions a) and c), while replacing b) by the condition:

b') the map  $F$ , considered on the boundary  $\partial W'^n$ , is a homotopy equivalence

$$F : \partial W'^n \rightarrow \partial W^n$$

of degree  $+1$ .

The answer to the problems posed depends on the dimension.

- 1)  $n = 2k + 1$ ; (P1) and (P2) are solved by the same means.
- 2)  $n = 4k$ ; in this case there arises an essential difficulty connected with the absence of an analogue of the Hirzebruch formula for manifolds with boundary. For (P1) this difficulty can be overcome; for (P2) it cannot.

3)  $n = 4k + 2$ ; (P1) and (P2) do not admit a complete solution, but in both cases results have been obtained which are, to the same extent, incomplete.

In what follows we shall consider only (P1), since (P2), in those cases where a solution can be obtained by the methods used here, is investigated analogously to (P1).

Let  $T_W$  denote the Thom space of the stable normal bundle of the manifold  $W^n$ , and let  $T_{\partial W}$  denote the Thom space of the stable normal bundle of the boundary. Obviously there is an embedding  $T_{\partial W} \subset T_W$ . Consider the relative homotopy group  $\pi_{N+n}(T_W, T_{\partial W})$  and the subset  $A$  of elements of  $\pi_{N+n}(T_W, T_{\partial W})$  defined as follows.

**Definition 1.** Denote the inverse image of the generating element  $e_{N+n}$  of the group  $H_{N+n}(T_W, T_{\partial W})$  under the Hurewicz homomorphism

$$H : \pi_{N+n}(T_W, T_{\partial W}) \rightarrow H_{N+n}(T_W, T_{\partial W})$$

by  $H^{-1}e_{N+n}$ . Define the set  $A$  as follows:  $\alpha \in A$  if there exists a smooth map of pairs

$$f : (D^{N+n}, S^{N+n-1}) \rightarrow (T_W, T_{\partial W}),$$

which is a representative of  $\alpha$ , is  $t$ -regular on  $W^n$  <sup>(8)</sup>, and is a diffeomorphism on  $f^{-1}(\partial W^n)$ . Clearly,  $A$  is a finite set.

**Definition 2.**  $\Pi^+ = \Pi(W^n, \partial W^n)$  is the group of homotopy classes of mappings of the pair  $(W^n, \partial W^n)$  onto itself such that for every  $g \in \Pi^+$  there exists a representative

$$F : (W^n, \partial W^n) \rightarrow (W^n, \partial W^n),$$

which is a smooth map of pairs, is a diffeomorphism on the boundary of degree +1, and for which the equality

$$F^*v_W = v_W$$

holds.

**Remark 1.** For case (P2) one must replace diffeomorphism on the boundary by homotopy equivalence.

**Definition 3.**  $\Pi_{SO} = \Pi(W^n, SO_N)$  is the group of homotopy classes of mappings  $W^n \rightarrow SO_N$ .

**Remark 2.** Consider the homomorphism

$$\partial : \pi_{N+n}(T_W, T_{\partial W}) \rightarrow \pi_{N+n-1}(T_{\partial W});$$

let  $C \subset \pi_{N+n-1}(T_{\partial W})$  be the set of homotopy classes of maps which have representatives

$$f : S^{N+n-1} \rightarrow T_{\partial W}$$

such that  $f/f^{-1}(\partial W^n)$  is a diffeomorphism. Then  $A = \partial^{-1}C$ .

The result of this paper is

**Theorem 1.** *The group  $\Pi_{SO}$  acts on the set  $A$ . The group  $\Pi^+$  acts on the set of orbits  $B = A/\Pi_{SO}$ .*

1) *If the dimension  $n \not\equiv 2 \pmod{4}$ , then the elements of the quotient set  $B/\Pi^+$  are in one-to-one correspondence with the set of smooth manifolds  $\{W'^n\}_1$ , considered up to a diffeomorphism of degree +1.*

2) *If  $n \equiv 2 \pmod{4}$ , then the set  $B/\Pi^+$  contains a subset whose elements are in one-to-one correspondence with the set of manifolds  $\{W'^n\}_1$ , considered up to diffeomorphism.*

**Remark 3.** Imposing the condition  $n \not\equiv 0 \pmod{4}$ , one can obtain a completely analogous theorem for case (P2).

For the proof of Theorem 1 we use the scheme developed in <sup>(5,6)</sup>, which applies without essential changes up to middle dimension. As is known, in this case one must perform a Morse reconstruction on the kernel of the homomorphism

$$F_* : H_i(W_F^n) \rightarrow H_i(W^n),$$

where  $F$  is a smooth  $t$ -regular map of pairs

$$F : (D^{N+n}, S^{N+n-1}) \rightarrow (H_W, T_{\partial W})$$

and  $W_F^n = F^{-1}(W^n)$ . As Browder showed <sup>(4)</sup>, the kernel can always be split off as a direct summand if the degree of the map  $F$  is +1, which in the present case holds because of the restrictions on the mapping.

Additional difficulties specific to manifolds with boundary appear in the middle dimension. Namely:

1. It is necessary to take into account that the kernel of the homomorphism is closed with respect to Poincaré duality.

**Lemma 1.** Let  $W_F^n$  and  $W^n$  be two smooth manifolds with boundaries  $\partial W_F^n$  and  $\partial W^n$ ,  $n > 4$ . Suppose there is a map of pairs

$$F : (W_F^n, \partial W_F^n) \rightarrow (W^n, \partial W^n),$$

satisfying the conditions:

- 1) The map  $F$ , considered on the boundary  $\partial W_F^n$ , is a homotopy equivalence

$$F | \partial W_F^n : \partial W_F^n \rightarrow \partial W^n.$$

- 2) The homomorphism

$$F_* : H_i(W_F^n) \rightarrow H_i(W^n)$$

is an isomorphism for  $i \leq [n/2] - 1$ , and in the case  $n = 2k + 1$  an isomorphism for  $i \leq [n/2]$  modulo torsion.

Then:

- a) the homomorphism

$$F_* : H_i(W_F^n, \partial W_F^n) \rightarrow H_i(W^n, \partial W^n)$$

is an isomorphism for  $i \leq [n/2] - 1$ ;

- b) the homomorphism  $F_*$  for absolute and relative homology in dimension  $[n/2]$  is an epimorphism;  
 c) the homomorphism

$$j_* : H_{[n/2]}(W_F^n) \rightarrow H_{[n/2]}(W_F^n, \partial W_F^n)$$

is monomorphic on the subgroup  $\text{Ker } F_*[H_{[n/2]}(W_F^n)]$ , and the kernel for relative homology  $\text{Ker } F_*[H_{[n/2]}(W_F^n, \partial W_F^n)]$  is naturally isomorphic to the kernel for absolute groups;

- d) the kernel is closed with respect to Poincaré duality.

- II. For  $n \equiv 0 \pmod{4}$  it is necessary to prove that the signature of the intersection matrix of the kernel is equal to zero. For manifolds with boundary there is no analogue of Hirzebruch's formula; however, the notion of index makes sense, namely, it is the index of the quadratic (generally speaking, degenerate) form defined on  $H_{2k}(W^{4k}, Q)$  by means of homological intersection. (Here  $Q$  denotes the rational numbers.)

**Lemma 2.** If there is a smooth manifold  $\omega^{4k+1}$  such that

$$\partial\omega^{4k+1} = W_1^{4k} \cup W_2^{4k},$$

where  $W_1^{4k}$  and  $W_2^{4k}$  are manifolds with common boundary  $V^{4k-1}$  such that  $\partial\omega^{4k+1}$  is obtained by gluing  $W_1^{4k}$  and  $W_2^{4k}$  along the boundary  $V^{4k-1}$  (all manifolds under consideration are oriented), then the indices of  $W_1^{4k}$  and  $W_2^{4k}$  are equal:

$$I(W_1^{4k}) = I(W_2^{4k}).$$

**Corollary.** If for two manifolds with boundary of dimension  $4k$ ,  $W_1^{4k}$  and  $W_2^{4k}$ , there exists a map

$$F : W_F^{4k} \rightarrow W^{4k},$$

satisfying the conditions (1), then  $I(W_F^{4k}) = I(W^{4k})$ .

**Remark 4.** There is no corresponding analogue for (2).

Together with Lemma 1, the corollary makes it possible to obtain the following assertion:

**Lemma 3.** Let  $W_F^{4k}$  and  $W^{4k}$  be the manifolds described in Lemma 1. Then the homomorphism

$$F_* : H_{2k}(W_F^{4k}) \rightarrow H_{2k}(W^{4k})$$

is a free abelian group, the intersection matrix of the kernel is unimodular and has signature equal to zero.

Lemmas 1 and 3 make it possible to carry out the necessary Morse modifications if  $n \not\equiv 2 \pmod{4}$ . In the latter case there arises an obstruction in the form of the Kervaire invariant <sup>(3,5)</sup>.

With respect to manifolds corresponding to one and the same element of  $A$ , the following holds.

**Lemma 4.** *If two manifolds  $W_1'^n$  and  $W_2'^n$  belong to one and the same element of  $A$  and are homotopically equivalent to the manifold  $W^n$ , then there exists a manifold with boundary  $\omega^{n+1}$  such that  $\partial\omega^{n+1} = W_1'^n \cup W_2'^n$ , and mappings  $F_1, F_2$*

$$F_1 : \omega^{n+1} \rightarrow W_1'^n,$$

$$F_2 : \omega^{n+1} \rightarrow W_2'^n,$$

which are retractions, with

$$F_1^* \nu_{W'_1} = \nu_\omega, \quad F_2^* \nu_{W'_2} = \nu_\omega.$$

The proof of this fact does not differ in any way from that given in paper <sup>(5)</sup>.

**Lemma 5.** *If two manifolds belong to the same class  $a \in A$ , then they are diffeomorphic.*

The proof of Lemma 5 uses Lemma 4 and a variant of Poincaré duality, which gives

**Lemma 6.** *Let there be a smooth manifold  $\omega^n$  with boundary  $\partial\omega$ , where  $\partial\omega = V_1 \cup V_2$ , and  $V_1$  and  $V_2$  are manifolds with boundary  $\partial V_1 = \partial V_2 = V_1 \cap V_2$ . All manifolds considered are oriented. Under these conditions there exists a natural Poincaré duality*

$$P : H_i(\omega, V_1) \approx H^{n-i}(\omega, V_2).$$

**Remark 5.** In the proof of Lemma 5 it is convenient to use the  $h$ -cobordism formulated in <sup>(2)</sup> and following from <sup>(2,7)</sup>.

As an application of the results obtained, one may indicate the following fact.

**Theorem 2.** *Consider a smooth manifold  $W^n$  with boundary  $\partial W^n$ , whose homology groups satisfy the condition*

$$H_{4k}(W^n) \otimes Q, \quad k = 1, 2, \dots,$$

$Q$  being the field of rational numbers. Then the set of smooth manifolds with boundary  $\partial W^n$  satisfying conditions a) and b) of problem (P1) and considered up to diffeomorphism is finite.

For the proof it suffices to observe that from the condition on the homology of  $W^n$  there follows the finiteness of stable  $SO(N)$ -bundles over  $W^n$ . This fact follows from consideration of the homotopy groups of the classifying space  $B_{SO(N)}$

$$\pi_i(B_{SO(N)}), \quad i < N,$$

and the isomorphism of the Hurewicz homomorphism on the homotopy groups

$$\pi_{4k}(B_{SO(N)}), \quad k = 1, 2, \dots$$

If  $\nu_W^{(1)}, \dots, \nu_W^{(m)}$  are the stable  $SO(N)$ -bundles over  $W^n$ , then, considering for each of the triples

$$(W^n, \partial W^n, \nu_W^{(i)}), \quad i = 1, \dots, m$$

(where  $m$  is the number of bundles) problem (P1), we obtain the required result.

**Remark 6.** Case (P2) is analogous to (P1).

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*Note: Figure translations are in progress. See original paper for figures.*

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