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Abstract

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THE FREE PRODUCT OF ASSOCIATIVE CALCULI WITH AMALGAMATED SUBALPHABET AND SOME RELATED QUESTIONS

(Presented by Academician P. S. Novikov on 27 IV 1964)

1. Let $\Gamma = A \cup B$, $H = A \cap B$, where A and B are alphabets. Let $\mathfrak{A}$ and $\mathfrak{B}$ be associative calculi[1], respectively, in the alphabets A and B. We shall assume that for every word M in H there is a word $[M^{-1}]$ in H such that

$$\mathfrak{A}: \; M[M^{-1}M] \parallel [M^{-1}M] \parallel \Lambda$$

(the group condition), and that for any words M and N in H, from

$$\mathfrak{A}: \; M \parallel N$$

there follows

$$\mathfrak{B}: \; M \parallel N$$

and conversely (the isomorphism condition). Then the associative calculus $\mathfrak{G}$ in the alphabet Γ, determined by the system obtained by uniting the defining systems of the calculi $\mathfrak{A}$ and $\mathfrak{B}$, is called the **free product of the associative calculi** $\mathfrak{A}$ and $\mathfrak{B}$ **with amalgamated subalphabet** H.

Theorem 1. *Let $\mathfrak{G}$ be the free product of the associative calculi $\mathfrak{A}$ and $\mathfrak{B}$ in the alphabets A and B, respectively, with amalgamated subalphabet H. Then, if P and Q are words in A and*

$$\mathfrak{G}: \; P \parallel Q,$$

then

$$\mathfrak{A}: \; P \parallel Q.$$

Theorem 2. *Under the same hypotheses, if P is a word in A, Q is a word in B, and*

$$\mathfrak{G}: \; P \parallel Q,$$

then one can find a word M in H such that

$$\mathfrak{A}: \; P \parallel M$$

and

$$\mathfrak{B}:\ Q \parallel M.$$

The method of proof is as follows. Introduce an auxiliary alphabet $D = \{p, q\}$ consisting of two letters not belonging to the alphabet Γ. A word W in the alphabet $\Pi = \Gamma \cup D$ is called a **proper word of order** ν, if it has the form

$$W = pK_1qL_1pK_2qL_2 \dots pK_\nu qL_\nu pK_{\nu+1}q,$$

where $\nu \geq 0$; $K_1, K_2, \dots, K_{\nu+1}$ are words in A; $L_1, L_2, \dots, L_\nu$ are words in B. A proper word W is called **perfect** if none of the words

$$L_1,\ K_2,\ L_2,\ \dots,\ K_\nu,\ L_\nu$$

is a word in H.

A sequence of $2\nu + 2$ numbers

$$S = \{s_0, s_1, \dots, s_{2\nu+1}\}$$

is called a **proper numbering of order** ν, if the following conditions are fulfilled.

a) The sequence S contains each of the numbers $0, 1, \dots, \nu$ exactly twice.

b) If $s_\alpha = s_\beta$, where $0 \leq \alpha < \beta \leq 2\nu + 1$, then the number $\beta - \alpha$ is odd.

c) There is no quadruple of numbers $\alpha, \beta, \gamma, \delta$ such that

$$0 \leq \alpha < \beta < \gamma < \delta \leq 2\nu + 1$$

and

$$s_\alpha = s_\gamma, \qquad s_\beta = s_\delta.$$

d) $s_0 = s_{2\nu+1} = 0$.

The following correspondence is introduced between the numbers of the numbering S and the occurrences of the letters of the alphabet D in a proper word W: the number s_0 is called the **number of the first occurrence** in W of the letter p, s_1 the number of the first occurrence in W of the letter q, and so on. Occurrences having the same number constitute a **pair**, and the common number of these occurrences is called the **number of the pair**. The pair with number i is denoted by ω_i.

Let $0 \leq \alpha < \beta \leq 2\nu + 1$ and $s_\alpha = s_\beta = i$. The pair ω_i is called a **pair of type** A (of type B) if α is even (odd). The word Ω_i is defined by putting, in the first case,

$$\Omega_i = pK_{\frac{\alpha+2}{2}}qL_{\frac{\alpha+2}{2}} \dots L_{\frac{\beta-1}{2}}pK_{\frac{\beta+1}{2}}q,$$

and in the second

$$\Omega_i = qL_{\frac{\alpha+1}{2}} pK_{\frac{\alpha+3}{2}} \cdots K_{\frac{\beta}{2}} qL_{\frac{\beta}{2}} p.$$

The words Ω_i $(i = 0, 1, \dots, \nu)$ occur in the word W, and $\Omega_0 = W$. Each of the words Ω_i can be represented in a unique way in the form

$$\Omega_i = pK_{t_1} \Omega_{u_1} L_{t_2} \cdots L_{t_r} \Omega_{u_r} L_{t_{r+1}}, \tag{1}$$

if ω_i is a pair of type A, or in the form

$$\Omega_i = qL_{t_1} \Omega_{u_1} L_{t_2} \cdots L_{t_r} \Omega_{u_r} L_{t_{r+1}}, \tag{2}$$

if ω_i is a pair of type B. Here $r \geq 0$; $t_1, t_2, \dots, t_{r+1}$ are some of the numbers $1, 2, \dots, \nu + 1$; $u_1, u_2, \dots, u_r$ are some of the numbers $1, 2, \dots, \nu$.

Suppose we are given a list of words $M_0, M_1, \dots, M_\nu$, where M_0 is a word in A, and $M_1, M_2, \dots, M_\nu$ are words in H. Construct the words $N_0, N_1, \dots, N_\nu$, putting, in accordance with (1) and (2),

$$N_i = K_{t_1} M_{u_1} K_{t_2} \cdots K_{t_r} M_{u_r} K_{t_{r+1}}, \tag{3}$$

if ω_i is a pair of type A, or

$$N_i = L_{t_1} M_{u_1} L_{t_2} \cdots L_{t_r} M_{u_r} L_{t_{r+1}}, \tag{4}$$

if ω_i is a pair of type B.

Let Q be a word in Γ. A system of objects $\{\nu, W, S, M_0, M_1, \dots, M_\nu\}$, where ν is a nonnegative integer, W is a proper word of order ν, S is a proper numbering of order ν, M_0 is a word in A, and $M_1, M_2, \dots, M_\nu$ are words in H, is called an **analysis** of the word Q if the following conditions hold:

A.1. $[W]^\Gamma = Q$.

A.2. For all $i = 0, 1, \dots, \nu$, $\mathfrak{A} : M_i \parallel N_i$, if ω_i is a pair of type A, or $\mathfrak{B} : M_i \parallel N_i$, if ω_i is a pair of type B, where the words N_i are defined according to (3) and (4).

The word M_0 is called the **basis** of the analysis. The analysis is called **complete** if the word W is complete.

Lemma 1. *If Q is a word in Γ, P is a word in A, and $\mathfrak{G} : P \parallel Q$, then the word Q has a complete analysis with basis P.*

The assertions of Theorems 1 and 2 are obtained from Lemma 1 as corollaries.

In the special case when $\mathfrak{A}$ and $\mathfrak{B}$ are inverse calculi (^{2}), the results stated can be regarded as an introduction to the constructive theory of the free product of groups with an amalgamated subgroup.

2. Consider the case in which $\mathfrak{A}$ and $\mathfrak{B}$ are inverse calculi and the alphabet H is empty. Then the group conditions and the isomorphism condition are satisfied automatically, and the calculus $\mathfrak{G}$ is called the **free product** of the inverse calculi $\mathfrak{A}$ and $\mathfrak{B}$. Let $\mathfrak{C} = \{C_1, C_2, \dots, C_m\}$ be some system of words in Γ. A word Q is called **dependent** in $\mathfrak{G}$ on $\mathfrak{C}$ if

$$\mathfrak{G}: \ Q \parallel C_{k_1}^{\varepsilon_1} C_{k_2}^{\varepsilon_2} \cdots C_{k_r}^{\varepsilon_r},$$

where $r = 0$, $1 \leq k_i \leq m$, $\varepsilon_i = \pm 1$, $i = 1, 2, \dots, r$.

The system of words $\mathfrak{C}$ is called a **system of generators** of the calculus $\mathfrak{G}$ if every word in Γ is dependent in $\mathfrak{G}$ on $\mathfrak{C}$.

Theorem 3 (constructive form of Grushko’s theorem). *Let $\mathfrak{G}$ be the free product of inverse calculi $\mathfrak{A}$ and $\mathfrak{B}$ in the alphabets A and B, respectively. Then, if the calculus $\mathfrak{G}$ has a system of generators consisting of m words, one can find a system of generators of the calculus $\mathfrak{G}$ consisting of no more than m words, each of which is a word in one of the alphabets A, B.*

A constructive proof of Theorem 3 was obtained by me on the basis of Lemma 1. Its plan was borrowed from Neumann’ s paper (^{3}).

3. Let $\mathfrak{A}$ be an inverse calculus in some alphabet A, and let

$$= \{b_1, b_2, \dots, b_p, b_1^{-1}, b_2^{-1}, \dots, b_p^{-1}\}$$

be an alphabet having no letters in common with A. Let $\mathfrak{G}$ be an inverse calculus in the alphabet $\Gamma = A \cup$, defined by the system

$$\begin{cases} U_i \leftrightarrow \\ b_{v_j}^{-1} A_j b_{v_j} \mid B_j^{-1} \leftrightarrow \end{cases}$$

$$(i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m),$$

where $\{U_i \leftrightarrow (i = 1, 2, \dots, n)\}$ is the defining system of the calculus $\mathfrak{A}$, $1 \leq v_j \leq p$, and A_j, B_j are words in A. Suppose that the following **isomorphism condition** is satisfied:

If

$$\mathfrak{A}: \ A_{k_1}^{\varepsilon_1} A_{k_2}^{\varepsilon_2} \cdots A_{k_r}^{\varepsilon_r} \parallel \Lambda,$$

then

$$\mathfrak{A}:\ B_{k_1}^{\varepsilon_1} B_{k_2}^{\varepsilon_2} \cdots B_{k_r}^{\varepsilon_r} \parallel \Lambda$$

and conversely, where $r \geq 1$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r = \pm 1$, and $k_1, k_2, \ldots, k_r$ are some of the numbers $1, 2, \ldots, m$. Then the letters $b_1, b_2, \ldots, b_p$ are called **stable letters** for the calculus $\mathfrak{G}$ with base $\mathfrak{A}$.

Theorem 4. Let the inverse calculus $\mathfrak{G}$ in the alphabet Γ have stable letters $b_1, b_2, \ldots, b_p$ with base $\mathfrak{A}$ in the alphabet A. Then, if Q is a word in A and

$$\mathfrak{G}:\ Q \parallel \Lambda,$$

then

$$\mathfrak{A}:\ Q \parallel \Lambda.$$

Theorem 5. Under the same assumptions, if Q is a word in Γ which is not a word in A, and

$$\mathfrak{G}:\ Q \parallel \Lambda,$$

then there is an occurrence in the word Q of some word of the form

$$b_{v_j}^{\varepsilon} C b_v^{-\varepsilon},$$

where $1 \leq v \leq p$, and C is a word in A depending on the system of words $\{A_1, A_2, \ldots, A_m\}$, if $\varepsilon = -1$, or on the system $\{B_1, B_2, \ldots, B_m\}$, if $\varepsilon = 1$.

For the proof of Theorems 4 and 5 one may apply a method analogous to the method used to prove Theorems 1 and 2.

4. Let $A = \{a_1, a_2, \ldots, a_m\}$ be some alphabet, and let $\mathfrak{B}$ be an associative calculus in the alphabet $\ = A \cup \{q\}$, defined by the system

$$\{F_{iq} \leftrightarrow qK_i\} \quad (i = 1, 2, \ldots, n),$$

where F_i, K_i are words in A. Words in having exactly one occurrence of the letter q will be called **special**. Let Γ be the alphabet consisting of the letters

$$a_1, a_2, \ldots, a_m,\ q,\ k,\ t,\ x,\ y,\ l_1, l_2, \ldots, l_n,\ r_1, r_2, \ldots, r_n$$

and their inverses. Let $\mathfrak{G}$ be the inverse calculus in the alphabet Γ, defined by the system

$$\begin{cases} yya_jy^{-1}a_j^{-1} \leftrightarrow \\ a_jxxa_j^{-1}x^{-1} \leftrightarrow \\ yl_iya_jl_i^{-1}a_j^{-1} \leftrightarrow \\ a_jxr_{ix}a_j^{-1}r_i^{-1} \leftrightarrow \\ l_iqK_ir_iq^{-1} \mid F_i^{-1} \leftrightarrow \\ l_itl_i^{-1}t^{-1} \leftrightarrow \\ yty^{-1}t^{-1} \leftrightarrow \\ kr_ik^{-1}r_i^{-1} \leftrightarrow \\ kxk^{-1}x^{-1} \leftrightarrow \\ kq^{-1}tqk^{-1}q^{-1}t^{-1}q \leftrightarrow \end{cases}$$

$$(i = 1, 2, \dots, n; \quad j = 1, 2, \dots, m).$$

Theorem 6. If Σ is a special word in , then from

$$\mathfrak{B} : \ \Sigma \parallel q$$

it follows that

$$\mathfrak{G} : \ k[\Sigma^{-1}t\Sigma k^{-1}[\Sigma^{-1}t^{-1}\Sigma \parallel \Lambda$$

and conversely.

Let $\mathfrak{A}$ be an associative calculus in the alphabet $A = \{a_1a_2, \dots, a_m\}$, defined by the system

$$\{T_i \leftrightarrow U_i\} \quad (i = 1, 2, \dots, l),$$

with an undecidable equivalence problem for the empty word (such a calculus was constructed

by A. A. Markov ([1]). Let $\mathfrak{B}$ be an associative calculus in the alphabet

$$B = A \cup \{q\},$$

defined by the system

$$\begin{cases} T_iq \leftrightarrow qU_j, \\ a_jq \leftrightarrow qa_j \end{cases}$$

$$(i = 1, 2, \dots, l; \ j = 1, 2, \dots, m).$$

Then, for the calculus $\mathfrak{B}$, the problem of equivalence of an arbitrary special word to the word q is undecidable.

Now Theorem 6 implies:

Theorem 7. *An inverse calculus* $\mathfrak{G}$ *with an undecidable equivalence problem can be constructed.*

The statements of Theorems 4–7, as well as the plan of the proofs of Theorems 6 and 7, are taken, with some modifications, from Britton' s paper ([4]). Thus a comparatively simple constructive proof has been obtained of P. S. Novikov' s result ([5]) on the "algorithmic undecidability of the identity problem" for finitely presented groups.

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Note: Figure translations are in progress. See original paper for figures.

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