HYDROMECHANICS

S. S. VOIT

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Abstract

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S. S. VOIT

WAVES EXCITED BY A PERIODIC SYSTEM OF PRESSURES MOVING OVER THE SURFACE OF A ROTATING LIQUID

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Let a liquid occupying the half-space $z < 0$ rotate with constant angular velocity ω about the vertical axis. We choose a rectangular coordinate system rotating together with the liquid about the axis Oz, and denote by u, v, w the components, relative to these axes, of the velocity of a particle passing at the instant t through the point (x, y, z). We write the linearized equations of motion in the form

$$\begin{gathered} \frac{\partial u}{\partial t} - 2\omega v = \frac{\partial P}{\partial x}, \\ \frac{\partial v}{\partial t} + 2\omega u = \frac{\partial P}{\partial y}, \\ \frac{\partial w}{\partial t} = \frac{\partial P}{\partial z}, \end{gathered} \tag{1}$$

where

$$P(x, y, z, t) = -\frac{1}{\rho}\left[p_0(x, y, t) + p_1(x, y, z, t)\right] - g\xi(x, y, t).$$

Here $p_1(x, y, z, t)$ is the dynamic part of the pressure due to the relative motion of the liquid particles, $p_0(x, y, t)$ is the pressure applied to the free surface of the liquid, and $\xi(x, y, t)$ is the deviation of the free surface of the liquid from its equilibrium state.

The system of equations (1), under the assumption of constant pressure $p_0(x, y, t)$, was used by Brillouin and Coulomb (^{1}) in the problem of the free and forced oscillations of a heavy liquid in a rotating circular cylinder. In contrast to the equations of motion of long waves, system (1) takes into account the vertical acceleration of the liquid particles.

The elevation of the free surface of the liquid is determined by the formula

$$\xi(x, y, t) = -\frac{1}{g}P(x, y, 0, t) - \frac{1}{\rho g}p_0(x, y, t). \tag{2}$$

The boundary condition for the function $P(x, y, z, t)$ has the form

$$\frac{\partial P(x,y,0,t)}{\partial z} + \frac{1}{g}\frac{\partial^2 P(x,y,0,t)}{\partial t^2} = -\frac{1}{\rho g}\frac{\partial^2 p_0(x,y,t)}{\partial t^2}. \tag{3}$$

The system of equations (1), taking into account the equation of continuity $\operatorname{div}\mathbf{v} = 0$, is reduced to the equation

$$\frac{\partial^2}{\partial t^2}\Delta P + 4\omega^2\frac{\partial^2 P}{\partial z^2} = 0, \tag{4}$$

where ΔP is the Laplace operator in the variables x, y, z.

We shall use equation (4) to solve the problem of waves excited by a system of pressures periodic in time and applied to the surface of the liquid in a region G, the region G moving over the free surface of the liquid with constant velocity c in the direction of the axis Ox. Put

$$p_0(x,y,t) = \bar{p}_0(x - ct, y)e^{i\sigma t} = \bar{p}_0(x', y)e^{i\sigma t}. \tag{5}$$

If the region G is a rectangle with sides $2a$ and $2b$, and the amplitude of the pressures throughout the region is constant and equal to p_0, then $\bar{p}_0(x', y)$

can be represented in the form

$$\bar{p}_0(x', y) = \frac{p_0}{\pi^2}\int_0^\infty \sin a\alpha\, e^{i\alpha x'}\,\frac{d\alpha}{\alpha}\int_{-\infty}^{\infty}\sin b\beta\, e^{i\beta y}\,\frac{d\beta}{\beta} +$$

$$+\frac{p_0}{\pi^2}\int_0^\infty \sin a\alpha\, e^{-i\alpha x'}\,\frac{d\alpha}{\alpha}\int_{-\infty}^{\infty}\sin b\beta\, e^{i\beta y}\,\frac{d\beta}{\beta}. \tag{6}$$

Putting $p_1(x,y,z,t) = \bar{p}_1(x',y,z)e^{i\sigma t}$, $\zeta(x,y,t) = \bar{\zeta}(x',y)e^{i\sigma t}$, $P(x,y,z,t) = \bar{P}(x',y,z)e^{i\sigma t}$, from equation (4) for the function $\bar{P}(x',y,z)$ we obtain

$$c^2\frac{\partial^2}{\partial x'^2}\Delta\bar{P} - 2i\sigma c\frac{\partial}{\partial x'}\Delta\bar{P} - \sigma^2\Delta\bar{P} + 4\omega^2\frac{\partial^2\bar{P}}{\partial z^2} = 0. \tag{7}$$

The boundary condition (3) is transformed to the form

$$g\frac{\partial\bar{P}}{\partial z} - \sigma^2\bar{P} - 2i\sigma c\frac{\partial\bar{P}}{\partial x'} + c^2\frac{\partial^2\bar{P}}{\partial x'^2} =$$

$$= -\frac{1}{\rho}\left[\frac{c^2\partial^2\bar{p}_0(x',y)}{\partial x'^2} - 2i\sigma c\frac{\partial\bar{p}_0(x',y)}{\partial x'} - \sigma^2\bar{p}_0(x',y)\right] \quad \text{for } z = 0. \tag{8}$$

Solving equation (7) with boundary condition (8) and using formula (2) for the elevation of the liquid, we obtain

$$\bar{\zeta}(x',y) = -\frac{\bar{p}_0(x',y)}{\rho g} - \frac{p_0}{\pi^2 \rho}\int_0^\infty (c\alpha-\sigma)\sqrt{(c\alpha-\sigma)^2-4\omega^2}$$

$$\times \frac{\sin a\alpha}{\alpha}\, e^{i\alpha x'}\, d\alpha \int_{-\infty}^{\infty} \frac{\sin b\beta\, e^{i\beta y}}{g\sqrt{\alpha^2+\beta^2}-(c\alpha-\sigma)\sqrt{(c\alpha-\sigma)^2-4\omega^2}}\,\frac{d\beta}{\beta}-$$

$$-\frac{p_0}{\pi^2\rho}\int_0^\infty (c\alpha+\sigma)\sqrt{(c\alpha+\sigma)^2-4\omega^2}\,\frac{\sin a\alpha}{\alpha}\, e^{-i\alpha x'}\, d\alpha$$

$$\times \int_{-\infty}^{\infty} \frac{\sin b\beta\, e^{i\beta y}}{g\sqrt{\alpha^2+\beta^2}-(c\alpha+\sigma)\sqrt{(c\alpha+\sigma)^2-4\omega^2}}\,\frac{d\beta}{\beta}. \tag{9}$$

Here, in the inner integrals of this formula, a definite rule is established for bypassing the poles, ensuring fulfillment of the radiation conditions.

We shall carry out an asymptotic analysis of the integrals of formula (9) for large values of the parameter $X = x/l$, where l is the characteristic size of the region G (for example, $l = a$), with bounded values of $|y|$.

Passing to integration with respect to the complex variable β, after first specifying the value of the root $\sqrt{\alpha^2+\beta^2}$ in the β-plane, we transform the inner integrals of formula (9) by methods of contour integration. As a result, for $x \gg a$ and $|y| < b$ we obtain

$$\zeta(x,y,t) = \frac{p_0 e^{i\sigma t}}{\pi\rho g}\int_0^\infty (c\alpha-\sigma)\sqrt{(c\alpha-\sigma)^2-4\omega^2}\,\frac{\sin a\alpha}{\alpha}\, e^{i\alpha x'}\times$$

$$\times\left\{\frac{1}{(c\alpha-\sigma)\sqrt{(c\alpha-\sigma)^2-4\omega^2}-g\alpha}\left[1-\cos(y\sqrt{F_1(\alpha)})e^{-ib\sqrt{F_1(\alpha)}}\right]\right.$$

$$\left.-\frac{1}{(c\alpha-\sigma)\sqrt{(c\alpha-\sigma)^2-4\omega^2}+g\alpha}\cos(y\sqrt{F_1(\alpha)})e^{-ib\sqrt{F_1(\alpha)}}\right\}d\alpha+$$

$$+\frac{p_0 e^{i\sigma t}}{\pi\rho g}\int_0^\infty (c\alpha+\sigma)\sqrt{(c\alpha+\sigma)^2-4\omega^2}\,\frac{\sin a\alpha}{\alpha}\, e^{-i\alpha x'}\times$$

$$\times\left\{\frac{1}{(c\alpha+\sigma)\sqrt{(c\alpha+\sigma)^2-4\omega^2}-g\alpha}\left[1-\cos\left(y\sqrt{F_2(\alpha)}\right)e^{-ib\sqrt{F_2(\alpha)}}\right]\right.$$

$$\left. - \frac{1}{(c\alpha + \sigma)\sqrt{(c\alpha+\sigma)^2 - 4\omega^2 + g\alpha}} \cos\left(y\sqrt{F_2(\alpha)}\right) e^{-ib\sqrt{F_2(\alpha)}} \right\} d\alpha + I_2. \quad (10)$$

Here

$$F_j(\alpha) = \frac{1}{g^2}(c\alpha \mp \sigma)^2 \left[(c\alpha \mp \sigma)^2 - 4\omega^2\right] - \alpha^2 \qquad (j = 1, 2). \quad (11)$$

The term I_2 contains double integrals whose integrands have no singularities in the domain of integration; they are easily estimated asymptotically, and it can be shown that they are of order not exceeding $1/x$. This term does not give wave elevations and characterizes the "deflection" of the free surface of the liquid, i.e., a certain oscillation of the free surface about the equilibrium position, decaying with distance from the region G.

Carrying out an analysis of the positions of the roots of the polynomials $F_j(\alpha)$ in the plane of the complex variable α, depending on the relation between the parameters σ, c, ω, and g, we determine the position of the branch points of the roots $\sqrt{F_j(\alpha)}$. Passing to integration in the plane of the complex variable α makes it possible to conclude that the leading terms of order $1/\sqrt{x}$ in the asymptotic expansion are determined by the branch points of the roots $\sqrt{F_j(\alpha)}$ located on the real axis. These terms are computed, and the final asymptotic representation for the elevation of the liquid $\zeta(x', y, t)$ has, for $\sigma > 2\omega$, the form

$$\zeta(x', y, t) = A e^{i(\sigma t - \alpha_1' x')} \frac{1}{\sqrt{x'}} + O\left(\frac{1}{x'}\right) \qquad (x' > 0); \quad (12)$$

$$\zeta(x', y, t) = \left(B e^{i\alpha_1 x'} + C e^{i\alpha_2 x'} + D e^{-i\alpha_1' x'}\right) \frac{e^{i\sigma t}}{\sqrt{|x'|}} + O\left(\frac{1}{x'}\right) \qquad (x' < 0), \quad (13)$$

where α_i and α_i' are the real roots of the polynomials $F_1(\alpha)$ and $F_2(\alpha)$, respectively.

For $\sigma < 2\omega$,

$$\zeta(x', y, t) = O\left(\frac{1}{x'}\right) \qquad (x' > 0); \quad (14)$$

$$\zeta(x', y, t) = \left[B_1 e^{i(\sigma t + \alpha_1 x')} + D_1 e^{i(\sigma t - \alpha_1' x')}\right] \frac{1}{\sqrt{|x'|}} + O\left(\frac{1}{x'}\right) \qquad (x' < 0). \quad (15)$$

The coefficients of the asymptotic expansion are known functions of the parameters σ, c, ω, g. The terms with coefficient A in formula (12) and C in formula

(13) are present under the condition that the speed of displacement of the region G is less than a certain critical value c_0.

Thus, in the case $\sigma > 2\omega$ and when the speed of displacement of the pressures is less than c_0, three systems of waves are formed behind the region G, of which two propagate away from the region G, i.e., in the direction opposite to the direction of displacement of the pressures, while one system propagates after the region, i.e., in the same direction in which the pressures are displaced. Ahead of the region G, only one system of waves propagates, in the direction in which the pressures are displaced.

If the region G moves with a speed exceeding the critical value c_0, then no waves are formed ahead of the region, while behind the region G only two systems of waves propagate in the direction opposite to the direction of displacement of the pressures.

In the case $\sigma < 2\omega$, irrespective of the speed of displacement of the pressures, no waves are formed ahead of the region G, while behind the region G two systems of waves are formed, of which one propagates in the direction of displacement of the pressures and the other in the opposite direction.

A number of special cases of the general formula obtained are of independent interest.

1. If the fluid is not rotating ($\omega = 0$), then the roots of the polynomials $F_i(\alpha)$ can be found explicitly, and as a result, after separating the real part, we obtain, retaining terms of order $1/\sqrt{x'}$:

$$\xi(x', y, t) = -\frac{2\sqrt{\pi}\rho b}{\pi\rho g c}\sqrt{\frac{g\Omega_1}{\sqrt{g^2 - 4\sigma c g}}}\sin\frac{a\Omega_1}{2c^2}\cos\left(\sigma t - \frac{x'\Omega_1}{2c^2} - \frac{\pi}{4}\right)\frac{1}{\sqrt{x'}} \quad (16)$$

for $x' > 0$ and $0 < c < c_0 = g/4\sigma$;

$$\begin{aligned}\xi(x', y, t) = -\frac{2\sqrt{\pi}\rho b}{\pi\rho g c}\Bigg\{&\sqrt{\frac{g\Omega_2}{\sqrt{g^2 + 4\sigma c g}}}\sin\frac{a\Omega_2}{2c^2}\cos\left(\sigma t + \frac{x'\Omega_2}{2c^2} - \frac{\pi}{4}\right)\\ &+\sqrt{\frac{g\Omega_3}{\sqrt{g^2 + 4\sigma c g}}}\sin\frac{a\Omega_3}{2c^2}\cos\left(\sigma t + \frac{x'\Omega_3}{2c^2} + \frac{\pi}{4}\right)\\ &+\sqrt{\frac{g\Omega_4}{\sqrt{g^2 + 4\sigma c g}}}\sin\frac{a\Omega_4}{2c^2}\cos\left(\sigma t - \frac{x'\Omega_4}{2c^2} - \frac{\pi}{4}\right)\Bigg\}\frac{1}{\sqrt{|x'|}}\end{aligned} \quad (17)$$

for $x' < 0$, $0 < c < c_0 = g/4\sigma$, where

$$\Omega_{1,4} = g - 2\sigma c \mp \sqrt{g^2 - 4\sigma c g}, \qquad \Omega_{2,3} = g + 2\sigma c \mp \sqrt{g^2 + 4\sigma c g}.$$

For $c > g/4\sigma$ there is no wave motion in front of the region G, and in formula (16) the single term should be deleted; likewise, in formula (17) the last term should be deleted.

As $c \to 0$, the roots $\alpha_3^2 = \Omega_3/2c^2$ and $\alpha_4^2 = \Omega_4/2c^2$ tend to infinity, and the last two terms of formula (17) are absent. This cannot be obtained by a direct limiting passage as $c \to 0$ in formula (17), since under this condition the asymptotic expression for the last two terms loses its meaning.

This problem was solved by the author in another way [2], and for the plane case by Kaplan [3].

2. For $\sigma = 0$ we have the case of the formation of waves by a steady system of pressures moving over the surface of a rotating fluid. As a result of the asymptotic analysis we obtain

$$\xi = O\left(\frac{1}{x'}\right) \qquad \text{for } x' > 0; \tag{18}$$

$$\xi = -\frac{\sqrt{2\pi}\rho b}{\pi\rho c\sqrt[4]{4\omega^2c^2+g^2}} \sin\left(\frac{a\sqrt{4\omega^2c^2+g^2}}{c^2}\right) \cos\left(x'\frac{\sqrt{4\omega^2c^2+g^2}}{c^2} + \frac{\pi}{4}\right) \frac{1}{\sqrt{|x'|}} + O\left(\frac{1}{x'}\right) \qquad \text{for } x' < 0. \tag{19}$$

If, in addition, $\omega = 0$, we obtain the ordinary ship waves.

3. Finally, let us note the case $c = 0$, when pressures periodic in time are applied to the surface of a rotating fluid in a fixed region G. For $\sigma > 2\omega$ the formulas for the elevation of the fluid coincide with the result obtained if, in the first terms of formulas (16) and (17), one passes to the limit as $c \to 0$. For a non-rotating fluid this problem was solved by L. N. Sretenskii [4]. Under the condition $\sigma < 2\omega$ the polynomials $F_i(\alpha)$ have no real roots; the consequence of this is the absence of radiation of waves from the region G. The process reduces only to a "deflection" of the free surface.

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Marine Hydrophysical Institute
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.