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Abstract

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On the First Boundary-Value Problem for Certain Differential Equations with Constant Coefficients

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Boundary-value problems for the string equation were considered in papers ([1–4]), and for the wave equation in ([5]).

In the present note the first boundary-value problem in ellipsoidal domains is considered for equations of the form

$$L_\lambda u \equiv Mu - \lambda Lu = h, \tag{1}$$

$$u\Big|_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma = \cdots = \frac{\partial^{s-1} u}{\partial n^{s-1}}\Big|_\Gamma = 0, \tag{2}$$

where λ is a numerical parameter, and M and L are homogeneous, formally self-adjoint differential operators with constant coefficients of order $2s$. Questions of the existence and uniqueness of a generalized solution of this problem are investigated. Necessary and sufficient conditions are established for the existence of a generalized solution. Subspaces of uniqueness of the generalized solution are also indicated.

Let Ω be an arbitrary ellipsoid with equation $1 - \sum_{i,j=1}^n a_{ij} x_i x_j = 0$.

Denote by $L_2^s(\Omega)$ (respectively $L_2^{-s}(\Lambda)$) the Hilbert space of all functions defined in Ω for which the integral

$$\int_\Omega (1-g)^s f^2 \, d\Omega \quad \left(\text{respectively } \int_\Omega (1-g)^{-s} f^2 \, d\Omega\right)$$

is finite, where $1 - g$ denotes the left-hand side of the equation of the ellipsoid.

It is easy to prove the following lemma:

Lemma 1. *Polynomials are dense in* $L_2^s(\Omega)$.

Corollary. *Since the operator of multiplication by the factor* $(1-g)^s$*, mapping* $L_2^s(\Omega)$ *into* $L_2^{-s}(\Omega)$*, is isometric, polynomials of the form* $(1-g)^s p$ *are dense in* $L_2^{-s}(\Omega)$.

On the set of all polynomials, dense in $L_2^s(\Omega)$, define the operator T_λ as follows:

$$T_\lambda p = L_\lambda (1-g)^s p.$$

The operator T_λ is symmetric and, for each fixed λ, the system of its polynomial eigenfunctions forms an orthonormal basis in the set of all polynomials ([6]), and by virtue of Lemma 1 this system is complete in $L_2^s(\Omega)$.

It is established that on the set P_0 of all polynomials of the form $Q(x) = (1-g)^s p(x)$, the inequality

$$\sum_{i=1}^{r} \int_\Omega \left(\frac{\partial^s Q}{\partial x_i^s} \right)^2 d\Omega \geq C \int_\Omega \frac{Q^2}{(1-g)^s} \, d\Omega, \qquad 1 \leq r \leq n. \tag{3}$$

Let $\overset{0}{W}{}^s_{r,2}(\Omega)$ be the Hilbert space of functions obtained by completing P_θ in the norm

$$\|Q\|^2_{\overset{0}{W}{}^s_{r,2}(\Omega)} = \sum_{i_1 \ldots i_s = 1}^{r} \left\| \frac{\partial^s Q}{\partial x_{i_1} \cdots \partial x_{i_s}} \right\|^2_{L_2(\Omega)}. \tag{4}$$

Using (3), it is easy to prove that

$$\overset{0}{W}{}^s_{r,2}(\Omega) \subset L_2^{-s}(\Omega). \tag{5}$$

A function $u(x) \in L_2(\Omega)$ will be called a **generalized solution of problem** (1), (2), with right-hand side $h(x) \in L_2^s(\Omega)$, if for any smooth function $\varphi(x)$ satisfying the boundary conditions (2), the equality

$$(u, L_\lambda \varphi) = (h, \varphi), \tag{6}$$

holds, where (,) is the ordinary scalar product in $L_2(\Omega)$.

Let us note that, in view of (5), the right-hand side of equality (6) has meaning for any function $h(x) \in L_2^s(\Omega)$.

It is easy to see that if λ is an eigenvalue[6] of the homogeneous boundary-value problem (1), (2) with eigenfunctions $Q_i^{(\lambda)}(x)$, then $q_i^{(\lambda)}(x) = (1-g)^{-s} Q_i^{(\lambda)}(x)$ are eigenfunctions of the operator T_λ corresponding to the zero eigenvalue. Otherwise zero is not an eigenvalue of T_λ. Denote by F_λ the subspace in $L_2^{-s}(\Omega)$

orthogonal to all eigenfunctions $Q_i^{(\lambda)}(x)$, and by N_λ the closure of the set $q_i^{(\lambda)}(x)$ in $L_2^s(\Omega)$.

Theorem 1. *The generalized solution of problem* (1), (2) *is unique in the subspace* F_λ.

If λ is not an eigenvalue of the homogeneous problem (1), (2), then the generalized solution of problem (1), (2) is unique in $L_2^{-s}(\Omega)$, since in this case the subspace F_λ coincides with $L_2^{-s}(\Omega)$.

Let $h(x) \in L_2^s(\Omega)$ and

$$h = h_0 + \sum_{k=1}^{\infty} b_k p_k^{(\lambda)},$$

where $h_0 \in N_\lambda$, and $p_k^{(\lambda)}$ are eigenfunctions of the operator T_λ corresponding to eigenvalues $\mu_k^{(\lambda)}$ different from zero.

Theorem 2. *For the existence of a generalized solution of problem* (1), (2) *from* $L_2^{-s}(\Omega)$, *it is necessary and sufficient that* $h(x)$ *be orthogonal to the subspace* N_λ *and that the series*

$$\sum_{k=1}^{\infty} \left(\frac{b_k}{\mu_k^{(\lambda)}} \right)^2 \tag{7}$$

converge.

Necessity. Let $u(x) \in L_2^{-s}(\Omega)$ be a generalized solution of problem (1), (2). Taking as $\varphi(x)$ in equality (6) the eigenfunctions $Q_i^{(\lambda)}(x)$ corresponding to the eigenvalue λ, we obtain

$$(u, L_\lambda Q_i^{(\lambda)}) = (h, Q_i^{(\lambda)}) = 0$$

or

$$(h, Q_i^{(\lambda)}) = \left\{ \frac{Q_i^{(\lambda)}}{(1-g)^s} \right\} = \{h, q_i^{(\lambda)}\} = 0, \quad \text{i.e. } h(x) \perp N_\lambda.$$

Since $u \in L_2^{-s}(\Omega)$, then $w = (1-g)^{-s} u \in L_2^s(\Omega)$ and, consequently, can be expanded in a series in the eigenfunctions of the operator T_λ:

$$w = w_0 + \sum_{k=1}^{\infty} a_k p_k^{(\lambda)}.$$

where $w_0 \in N_\lambda$. Substituting in (6), in place of φ, the functions $(1-g)^s p_k^{(\lambda)}$, we obtain

$$(u, L_\lambda (1-g)^s p_k^{(\lambda)}) = \{w, T_\lambda p_k^{(\lambda)}\} = \mu_k^{(\lambda)} \{w, p_k^{(\lambda)}\} =$$

$$= \mu_k^{(\lambda)} a_k = (h, (1-g)^s p_k^{(\lambda)}) = \{h, p_k^{(\lambda)}\} = b_k,$$

whence $b_k/\mu_k^{(\lambda)} = a_k$, and, consequently, the series (7) converges.

Sufficiency. Let $h(x) \perp N_\lambda$ and let the series (7) converge; then

$$w = \sum_{k=1}^{\infty} \frac{b_k}{\mu_k^{(\lambda)}} p_k^{(\lambda)}$$

belongs to $L_2^s(\Omega)$. We shall show that $u = (1-g)^s w$ is a generalized solution of problem (1), (2).

We have

$$(u, L_\lambda \varphi) = \{w, L_\lambda \varphi\} = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{b_k}{\mu_k^{(\lambda)}} p_k^{(\lambda)} L_\lambda \varphi \right\} =$$

$$= \lim_{n\to\infty} \left(L_\lambda (1-g)^s \sum_{k=1}^{n} \frac{b_k}{\mu_k^{(\lambda)}} p_k^{(\lambda)}, \varphi \right) \lim_{n\to\infty} \left(T_\lambda \sum_{k=1}^{n} \frac{b_k}{\mu_k^{(\lambda)}} p_k^{(\lambda)}, \varphi \right) =$$

$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} b_k p_k^{(\lambda)}, \frac{\varphi}{(1-g)^s} \right\} = \left\{ h, \frac{\varphi}{(1-g)^s} \right\} = (h, \varphi),$$

i.e. $u(x)$ is a generalized solution of problem (1), (2), and, since $w \in L_2^s(\Omega)$, it follows that $u \in L_2^{-s}(\Omega)$.

Corollary. Let $h(x)$ be a polynomial; then

$$h(x) = \sum_{k=1}^{m} b_k p_k^{(\lambda)} + h_0$$

and the series (7) converges; consequently, a generalized solution of problem (1), (2) exists, moreover in the form of a polynomial, if $h(x) \perp N_\lambda$.

Let, in equation (1),

$$L = (-1)^s \sum_{i_1 \dots i_s,\, j_1 \dots j_s = 1}^{r} \beta_{i_1 \dots i_s,\, j_1 \dots j_s} \frac{\partial^{2s}}{\partial x_{i_1} \cdots \partial x_{i_s} \partial x_{j_1} \cdots \partial x_{j_s}}, \qquad 1 \le r \le n,$$

where for all real numbers $\xi_{i_1 \dots i_s}$ the inequality

$$\sum_{i_1 \dots i_s,\, j_1 \dots j_s = 1}^{r} \beta_{i_1 \dots i_s,\, j_1 \dots j_s} \xi_{i_1 \dots i_s} \xi_{j_1 \dots j_s} \ge \gamma^2 \sum_{i_1 \dots i_s = 1}^{r} \xi_{i_1 \dots i_s}^2. \tag{8}$$

holds. From inequality (3) it follows easily that the operator $T = L(1-g)^s$, considered on the set of all polynomials P, is positive definite in the scalar product $L_2^s(\Omega)$; therefore, repeating the arguments of (6), we obtain that the polynomial eigenfunctions $Q_i(x)$ of the homogeneous problem (1), (2) form a basis in P_0, and these polynomials may be chosen so that the relation

$$(LQ_i, Q_j) = [Q_i, Q_j] = \delta_{ij} \tag{9}$$

is satisfied.

Since the norm defined by (9) is equivalent to the norm of the space $\overset{0}{W}{}^s_{r,2}(\Omega)$, it follows that

Theorem 3. The polynomial eigenfunctions $Q_i(x)$ of the homogeneous problem (1), (2) form a complete orthonormal system in the scalar product (9) in $\overset{0}{W}{}^s_{r,2}(\Omega)$.

We now consider the equation

$$Lu = h. \tag{1*}$$

Lemma 2. If L satisfies condition (8), then problem (1*), (2), for any $h(x) \in L_2^s(\Omega)$, has a unique solution in $\overset{0}{W}{}^s_{r,2}(\Omega)$.

Indeed, from the positive definiteness of the operator $T = L(1-g)^s$ it follows that, for any $h \in L_2^s(\Omega)$, the series (7) converges, and, by Theorems 1, 2,

there exists a unique solution

$$u = \sum_{k=1}^{\infty} \frac{b_k}{\mu_k} (1-g)^s p_k$$

from $L_2^{-s}(\Omega)$. It is easy to show that the system of functions $\dfrac{(1-g)^s p_k}{\sqrt{\mu_k}}$ is orthonormal in the scalar product (9); consequently, $u \in \overset{0}{W}{}^s_{r,2}(\Omega)$.

Remark. From the equalities

$$\{h, p\} = (h, (1-g)^s p) = (u, L(1-g)^s p) = [u, (1-g)^s p]$$

it follows that if h is orthogonal to some polynomial p in $L_2^s(\Omega)$, then the generalized solution of problem (1*), (2) is orthogonal to the polynomial $(1-g)^s p$ in the scalar product (9).

Let us now consider problem (1), (2), assuming condition (8) to be satisfied. Denote by G_λ the closure, in the scalar product (9), of the set of eigenfunctions $Q_i^{(\lambda)}$ of this problem corresponding to the eigenvalue λ.

Let $h \perp N_\lambda$ and let $v(x) \in \overset{0}{W}{}^s_{r,2}(\Omega)$ be a generalized solution of problem (1*), (2). In view of the remark just made, we may assert that

$$v(x) = \sum_{k=1}^{\infty} v_k Q_k,$$

where Q_k are eigenfunctions of problem (1), (2) with eigenvalues λ_k different from λ.

Theorem 4. *For the existence of a generalized solution of problem (1), (2) from $\overset{0}{W}{}^s_{r,2}(\Omega)$, it is necessary and sufficient that the series*

$$\sum_{k=1}^{\infty} \left(\frac{v_k}{\lambda_k - \lambda} \right)^2 \tag{10}$$

converge.

In this case the solution is unique in the orthogonal complement to G_λ.

Remark. If some linear combination $\alpha M + \beta L$ of the operators M and L satisfies the positive-definiteness condition (8) with respect to the first $m > r$ variables, then in Theorems 3 and 4 $\overset{0}{W}{}^s_{r,2}(\Omega)$ may be replaced by the space $\overset{0}{W}{}^s_{m,2}(\Omega)$.

Finally, using Theorems 2, 4 and relation (5), we note that from the convergence of series (10) it is easy to conclude the convergence of series (7). However, one can construct an example showing that the converse assertion is false, i.e. the boundary-value problem (1), (2) may have a generalized solution from $L_2^{-s}(\Omega)$ which does not belong to $\overset{0}{W}{}^s_{r,2}(\Omega)$.

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Note: Figure translations are in progress. See original paper for figures.