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Abstract

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CORRESPONDENCES OVER A QUASI-EXACT CATEGORY

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1. In the paper ([1]) an axiomatic characterization was given of the category of correspondences over an Abelian category. The principal role in this characterization is played by the notion of a category with involution, or an I-category (for all definitions see §§ 2, 3). It was shown that the subcategory of all proper mappings of an I-category satisfying axioms (K1)–(K3) (see § 2) is quasi-exact. In the present note, for each quasi-exact category K, an I-category $\mathfrak{R}(K)$ of correspondences over K is constructed which satisfies axioms (K1)–(K3). The subcategory of all proper mappings of the category $\mathfrak{R}(K)$ turns out to be isomorphic to the category K.

2. Let us recall some definitions (see ([1])). A category $\mathfrak{R}$ is called a **category with involution** or an **I-category** if the following conditions are satisfied:

a) the set of mappings $\mathfrak{R}(a, b)$ of an object a into an object b, for any objects $a, b \in \mathfrak{R}$, is partially ordered by the relation $\subset$;

b) if $\alpha_1, \alpha_2 \in \mathfrak{R}(a, b)$, $\beta \in \mathfrak{R}(b, c)$, and $\alpha_1 \subset \alpha_2$, then $\alpha_1\beta \subset \alpha_2\beta$;

c) for any objects a, b there is given a mapping of the set $\mathfrak{R}(a, b)$ into the set $\mathfrak{R}(b, a)$, called **involution** ($\alpha \in \mathfrak{R}(a, b)$ goes into $\alpha^* \in \mathfrak{R}(b, a)$), with the following properties: c1) $(\alpha\beta)^* = \beta^*\alpha^*$; c2) $\alpha^{**} = \alpha$; c3) from $\alpha_1 \subset \alpha_2$ it follows that $\alpha_1^* \subset \alpha_2^*$ (for any mappings $\alpha, \alpha_1, \alpha_2 \in \mathfrak{R}(a, b)$, $\beta \in \mathfrak{R}(b, c)$).

We now formulate axioms (K1)–(K3), concerning I-categories.

(K1). In the I-category $\mathfrak{R}$ there exists an object 0 such that the set $\mathfrak{R}(0, 0)$ consists of the identity mapping ε_0, and the sets $\mathfrak{R}(0, a)$, for any object $a \in \mathfrak{R}$, have a greatest and a least element Ω_a and ω_a, respectively.

For a mapping $\alpha : a \to b$, put $I\alpha = \omega_a\alpha$, $B\alpha = \Omega_a\alpha$, $K\alpha = \omega_b\alpha^*$, $D\alpha = \Omega_b\alpha^*$. The mapping α is called **I-regular** (**B-, K-, D-regular**) if $I\alpha = \omega_b$ ($B\alpha = \Omega_b$, $K\alpha = \omega_a$, $D\alpha = \Omega_a$); α is called a **proper** mapping if it is I-regular and D-regular. A proper mapping is called a **projection** if it is B-regular, and is called an **injection** if it is K-regular.

(K2). If $\alpha \in \mathfrak{R}(a, b)$, $\beta \in \mathfrak{R}(c, b)$, then: a) from $I\alpha \subset I\beta$ it follows that $\beta\alpha^*\alpha \subset \beta$; b) from $B\alpha \supset B\beta$ it follows that $\beta\alpha^*\alpha \supset \beta$.

(K3). For any mapping $\alpha \in \mathfrak{R}(0, a)$, where a is an arbitrary object of $\mathfrak{R}$, there exist an injection $\mu : u \to a$ and a projection $\nu : a \to v$ such that $B\mu = \alpha$, $K\nu = \alpha$.

3. Let us recall the definition of a quasi-exact category ([1]), and for the terminology see ([2]). A category K is called **quasi-exact** if the following conditions are satisfied: 1) every mapping has a kernel and a cokernel; 2) every mapping is representable in the form of a product of a normal epimorphism and a normal monomorphism; 3) the subobjects of any object form a set.

Everywhere in what follows we shall be speaking of a fixed quasi-exact category K. In contrast to ([2]) (see ([1])), we shall assume that in each class of equivalent pairs of the form (u, μ), where $\mu : u \to a$ is a monomorphism, there has been chosen

representative, called a subobject of the object a. An analogous supposition is made about factor-objects. Let now $\mu_1 : u_1 \to a$ and $\mu_2 : u_2 \to a$ be two monomorphisms. A monomorphism μ will be called the **intersection** of the monomorphisms μ_1 and μ_2, $\mu = \mu_1 \cap \mu_2$, if $\mu = \mu_1' \mu_1 = \mu_2' \mu_2$ and if every monomorphism $\bar{\mu}$, representable in the form $\bar{\mu} = \bar{\mu}_1 \mu_1 = \bar{\mu}_2 \mu_2$, is representable in the form $\bar{\mu} = \mu' \mu$. The monomorphism μ is determined up to multiplication on the left by an invertible mapping. Let now (u_1, μ_1) and (u_2, μ_2) be subobjects of the object a, and let (u, μ) be their intersection. The monomorphism μ is the intersection of μ_1 and μ_2 in the sense of the definition just given, and therefore the subobject (u, μ) will be denoted by $(u_1 \cap u_2, \mu_1 \cap \mu_2)$. In what follows one must also bear in mind the dual definitions.

(Figure: Fig. 1)

Fig. 1

4. Let K be a quasi-exact category. Consider the class of all triples of the form
$$u \xrightarrow{\nu} x \xleftarrow{\nu'} u',$$
where ν, ν' are epimorphisms. In this class introduce an equivalence relation: the triple
$$u \xrightarrow{\nu} x \xleftarrow{\nu'} u'$$
is regarded as equivalent to the triple
$$v \xrightarrow{\pi} y \xleftarrow{\pi'} v',$$
if there exists an invertible mapping $\xi : x \to y$ such that $\nu\xi = \pi$, $\nu'\xi = \pi'$. A class of equivalent triples $\bar{\alpha}$ with representative
$$u \xrightarrow{\nu} x \xleftarrow{\nu'} u'$$
will be called a **correspondence** of the object a with the object b over the category K, if there exist such monomorphisms $\mu : u \to a$, $\mu' : u' \to a$, that the pairs (u, μ), (u', μ') are subobjects of the objects a and b, respectively (notation:
$$\bar{\alpha} = \langle u \xrightarrow{\nu} x \xleftarrow{\nu'} u', \mu, \mu' \rangle).$$

The correspondences of the object a with the object b, as is easy to see, form a set, which we shall denote by $\mathfrak{R}(a,b)$. In this set introduce a **partial order**: the correspondence

$$\bar{\alpha} = \langle u \xrightarrow{\nu} x \xleftarrow{\nu'} u', \mu, \mu' \rangle$$

precedes the correspondence

$$\bar{\beta} = \langle v \xrightarrow{\pi} y \xleftarrow{\pi'} v', \sigma, \sigma' \rangle,$$

$\bar{\alpha}, \bar{\beta} \in \mathfrak{R}(a,b)$, $\bar{\alpha} \subset \bar{\beta}$, if $(u,\mu) \leqslant (v,\sigma)$, $(u',\mu') \leqslant (v',\sigma')$, i.e. $\mu = \mu_1\sigma$, $\mu' = \mu_2'\sigma'$, and there exists a mapping $\varphi : x \to y$ such that $\nu\varphi = \mu_1\pi$, $\nu'\varphi = \mu_1'\pi'$.

If

$$\bar{\alpha} = \langle u \xrightarrow{\nu} x \xleftarrow{\nu'} u', \mu, \mu' \rangle \in \mathfrak{R}(a,b),$$

then set

$$\bar{\alpha}^* = \langle u' \xrightarrow{\nu'} x \xleftarrow{\nu} u, \mu', \mu \rangle \in \mathfrak{R}(b,a).$$

The indicated mapping of the set $\mathfrak{R}(a,b)$ into the set $\mathfrak{R}(b,a)$ will be called **involution**.

Let now

$$\bar{\alpha} = \langle u \xrightarrow{\nu} x \xleftarrow{\nu'} u', \mu, \mu' \rangle \in \mathfrak{R}(a,b), \qquad \bar{\beta} = \langle v \xrightarrow{\pi} y \xleftarrow{\pi'} v', \sigma, \sigma' \rangle \in \mathfrak{R}(b,c).$$

Put

$$(u',\mu') \cap (v,\sigma) = (u' \cap v, \mu' \cap \sigma).$$

Then $\mu' \cap \sigma = \mu_1\mu' = \sigma_1\sigma$. Let

$$\mu_1\nu' = \rho_1\tau_1, \qquad \sigma_1\pi = \rho_2\tau_2,$$

where $\rho_1\rho_2$ are epimorphisms, τ_1, τ_2 are monomorphisms. Choose such maximal subobjects

$$(\bar{u},\bar{\mu}) \leqslant (u,\mu), \qquad (\bar{v},\bar{\sigma}) \leqslant (v',\sigma')$$

of the objects a, c, respectively, that

$$\mu_1\nu = \lambda_1\tau_1, \qquad \sigma_1\pi' = \lambda_2\tau_2,$$

where $\bar{\mu}_1\mu = \bar{\mu}$, $\bar{\sigma}_1\sigma' = \bar{\sigma}$. It is known that the subobjects $(\bar{u},\bar{\mu})$, $(\bar{v},\bar{\sigma})$ are uniquely determined and λ_1, λ_2 are epimorphisms. If

$\rho = \rho_1 \cap \rho_2 : u' \cap v \to z$, $\rho = \rho_1\rho' = \rho_2\rho''$, then the correspondence $\tilde{\gamma} = \langle \bar{u} \xleftarrow{\lambda_1\rho'} z \xrightarrow{\lambda_2\rho''} \bar{v},\ \mu, \sigma \rangle$ will be called the **product** of the correspondences $\bar{\alpha}$ and $\bar{\beta}$: $\bar{\gamma} = \bar{\alpha} \circ \bar{\beta}$. It is not difficult to verify that $\bar{\gamma}$ does not depend on the choice of representatives of the correspondences $\bar{\alpha}, \bar{\beta}$, on the choice of representations of the mappings $\mu_1\nu', \sigma_1\pi$ in the form of a product of an epimorphism and a monomorphism, or on the choice of the epimorphism ρ.

In constructing the correspondence $\bar{\gamma}$, a commutative diagram arises; Fig. 1 makes this construction clear.

Theorem. *Correspondences over a quasiexact category K form an I-category $\mathfrak{K}(K)$, with respect to the multiplication, partial order, and involution introduced above, satisfying axioms (K1)–(K3). The category K is isomorphically embedded in the category $\mathfrak{K}(K)$ as the subcategory of all proper mappings.*

From this theorem and the results of [1] it follows that between quasiexact categories and I-categories in which axioms (K1)–(K3) are satisfied there is a one-to-one correspondence, under which each I-category corresponds to the subcategory of all its proper mappings, and conversely.

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CITED LITERATURE

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2. A. G. Kurosh, A. Kh. Livshits, E. G. Shulgeifer, *UMN*, **15**, No. 6 (96), 3 (1960).

Note: Figure translations are in progress. See original paper for figures.