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Abstract

Full Text

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A GENERAL BOUNDARY-VALUE PROBLEM WITH SHIFT FOR AN EQUATION OF ELLIPTIC TYPE OF SECOND ORDER

(Presented by Academician I. N. Vekua, 23 XII 1963)

§ 1. Let S^+ be a finite domain of the plane $z = x + iy$, bounded by a simple closed Lyapunov contour Γ. We shall assume that the positive direction of Γ leaves S^+ on the left. Suppose that the function $\alpha(t)$ homeomorphically maps the contour Γ onto itself, preserving the direction of traversal, has derivative $\alpha'(t) \in H$, different from zero everywhere on Γ, and, for some fixed natural number n,

$$\alpha_n(t) \equiv \alpha[\alpha_{n-1}(t)] = t \quad (\alpha_0(t) \equiv t,\ t \in \Gamma). \tag{1,1}$$

Consider the differential equation

$$\Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0, \tag{1,2}$$

where a, b, and c are real analytic functions of their arguments in some domain of definition of equation (1,2). In what follows we shall assume that the origin of coordinates lies in S^+ and that $S^+ \subset S_1^+$, where S_1^+ is the principal domain of equation (1,2) (see ([1])).

Problem $A(\alpha_n)$**.** Let m be some natural number or zero. It is required to find a real regular solution $u(x,y)$ of equation (1,2), continuous together with its derivatives of order m in $S^+ + \Gamma$ and satisfying on Γ the condition H, according to the boundary condition:

$$\sum_{\nu=0}^{n-1} \sum_{j,k=0}^{j+k\le m} \left\{ a_\nu^{j,k}(t_0)\, u_{j,k}^+[\alpha_\nu(t_0)] + \int_\Gamma b_\nu^{j,k}(t_0,\tau)\, u_{j,k}^+(\tau)\, d\sigma \right\} = f(t_0)$$

$$\left(u_{j,k}^+(t) = \left(\frac{\partial^{j+k} u}{\partial x^j \partial y^k} \right)^+ \right), \tag{1,3}$$

where $a_\nu^{j,k}(t_0)$, $f(t_0)$, and $b_\nu^{j,k}(t_0,\tau)$ are given real functions, with $a_\nu^{j,k}(t_0)$ and $f(t_0)$ belonging to the class H, while $b_\nu^{j,k}(t_0,\tau)$ have the form

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$$\tilde{b}_\nu^{j,k}(t_0,\tau) = |t_0-\tau|^\gamma b_\nu^{j,k}(t_0,\tau), \qquad \tilde{b}_\nu^{j,k}(t_0,\tau)\in H, \quad 0\le\gamma<1. \tag{1,4}$$

Let us first consider the case $m \geq 1$. Using the method of I. N. Vekua (see (^{1})), any solution of problem $A(\alpha_n)$ can be represented in the form

$$u(x,y) = \int_\Gamma \widetilde{K}(z,t)\,\mu(t)\,dt, \tag{1,5}$$

where

$$\widetilde{K}(z,t) = G(0,0,z,\bar z) + \operatorname{Re}\left\{G(z,0,z,\bar z)\left(1-\frac{z}{t}\right)^{m-1}\ln\left(1-\frac{z}{t}\right) - \right.$$
$$\left. - \int_0^z \left(1-\frac{\sigma}{t}\right)^{m-1}\ln\left(1-\frac{\sigma}{t}\right)\frac{\partial G(\sigma,0,z,\bar z)}{\partial\sigma}\,d\sigma\right\}; \tag{1,6}$$

where

$G(z,\xi,\tau,t)$ is the Riemann function of equation (1.2), $\mu(t)\in H$ is a real function which, for a given solution of problem $A(\alpha_n)$, is determined uniquely and satisfies the singular integral equation with shift

$$T\mu \equiv \sum_{\nu=0}^{n-1}\left\{A_\nu(t_0)\mu[\alpha_\nu(t_0)] + \frac{1}{\pi i}\int_\Gamma \frac{K_\nu(t_0,t)\mu(t)\,dt}{t-\alpha_\nu(t_0)}\right\} = f(t_0); \tag{1,7}$$

$$A_\nu(t_0) = \operatorname{Re}\left\{\pi i(-1)^m(m-1)!\,\alpha_\nu'^{-m}(t_0)\,\overline{\alpha_\nu'(s_0)}\,H_0[\alpha_\nu(t_0)]\sum_{k=0}^m i^k a_\nu^{m-k,k}(t_0)\right\}$$
$$\left(\alpha_\nu'(s) = \frac{d\alpha_\nu(t)}{ds_\nu}\right), \tag{1,8}$$

$$2K_\nu(t_0,t) = (-1)^m(m-1)!\pi i\left\{\bar t'\,t^{-m}H_0[\alpha_\nu(t_0)]\sum_{k=0}^m i^k a_\nu^{m-k,k}(t_0)\right.$$
$$+\,\bar t'\exp(2i\arg[t-\alpha_\nu(t_0)])\times$$
$$\left.\times\,\overline{t'^{-m}H_0[\alpha_\nu(t_0)]}\sum_{k=0}^m i^k a_\nu^{m-k,k}(t_0) + [t-\alpha_\nu(t_0)]\psi_\nu^*(t_0,t)\right\} \tag{1,9}$$

$$(H_0(t_0) = G(t_0,0,t_0,\bar t_0));$$

$\psi_\nu^*(t_0, t)$ are quite definite functions which, when $t = \alpha_\nu(t_0)$, may have a singularity only of logarithmic type. It is evident that $A_\nu(t_0)$, $K_\nu(t_0, t) \in H$ $(\nu = 0, 1, \dots, n-1)$.

Equation (1.7) is equivalent to problem $A(\alpha_n)$; in particular, the homogeneous problem $A^0(\alpha_n)$ $(f = 0)$ is equivalent to the homogeneous equation $T\mu = 0$; linearly independent (over the field of real numbers) solutions of problem $A^0(\alpha_n)$ correspond to linearly independent solutions of the homogeneous equation $T\mu = 0$, and conversely.

§ 2. Consider a singular integral equation of the form (1.7), where $A_\nu(t_0)$, $K_\nu(t_0, t)$, and $f(t_0)$ are prescribed functions (generally speaking, complex-valued) of class H. The adjoint operator T' of the operator T is given by the formula

$$T'\mu \equiv \sum_{\nu=0}^{n-1} \left\{ \alpha_\nu'(t_0) A_{n-\nu}[\alpha_\nu(t_0)] \psi[\alpha_\nu(t_0)] - \frac{1}{\pi i} \int_\Gamma \frac{K_{n-\nu}(t, t_0)\psi(t)\, dt}{\alpha_{n-\nu}(t) - t_0} \right\}$$

$$(A_n[\,] \equiv A_0[\,] \equiv K_n[\,] \equiv K_0[\,]). \tag{2,1}$$

Let $b_0(t), b_1(t), \dots, b_{n-1}(t)$ be arbitrary functions of a point of the contour Γ. We shall agree to denote by $R[b_0(t), \dots, b_{n-1}(t)]$ the matrix formed from the elements $b_0, b_1, \dots, b_{n-1}$ according to the following rule:

$$R[b_0(t), \dots, b_{n-1}(t)] \equiv \begin{vmatrix} b_0(t) & b_1(t) & \cdots & b_{n-1}(t) \\ b_{n-1}[\alpha(t)] & b_0[\alpha(t)] & \cdots & b_{n-2}[\alpha(t)] \\ \cdot & \cdot & \cdot & \cdot \\ b_1[\alpha_{n-1}(t)] & b_2[\alpha_{n-1}(t)] & \cdots & b_0[\alpha_{n-1}(t)] \end{vmatrix}. \tag{2,2}$$

We shall call the equation $T\mu = f$ an equation of **normal type** if

$$\det\{R[A_0(t_0), \dots, A_{n-1}(t_0)] \pm$$

$$\pm R[K_0(t_0, t_0), K_1[t_0, \alpha(t_0)], \dots, K_{n-1}[t_0 \alpha_{n-1}(t_0)]]\} \neq 0. \tag{2,3}$$

Theorem 1. *Condition (2.3) is necessary and sufficient for the validity of the assertions: a) the equation $T\mu = 0$ has a finite number k of linearly independent solutions; b) for the solvability of the equation $T\mu = f$ it is necessary and sufficient that there be a finite number k' of conditions of the form*

$$\int_\Gamma \psi_j(t) f(t)\, dt = 0,$$

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where $\psi_j(t)$ $(j = 1, 2, \dots, k')$ *is a complete system of linearly independent solutions of the adjoint equation* $T'\psi = 0$.

Linear independence in this paragraph is understood over the field of complex numbers.

Denote by $D(T)$ the range of the operator T. We shall say that the equation $T\mu = f$ is **solvable up to finite-dimensional subspaces** if the quotient space $H/D(T)$ is finite-dimensional.

Theorem 2. *For the solvability of the equation* $T\mu = f$ *up to finite-dimensional subspaces, it is necessary and sufficient that condition (2.3) be satisfied.*

The sufficiency of the condition of Theorem 1 was proved by us earlier (5). From the results of the same work, from the corresponding theorems of I. Ts. Gokhberg (2,3) and A. I. Volpert (4), the second part of Theorem 1 and Theorem 2 follow.

§ 3. Let us return to the problem $A(\alpha_n)$. We shall say that the problem $A(\alpha_n)$ is **solvable up to finite-dimensional subspaces** if the singular integral equation with shift (1.7) equivalent to it is solvable up to finite-dimensional subspaces.

Taking into account the results of §§ 1 and 2, it is not difficult to prove the validity of the following theorem:

Theorem 3. *The condition*

$$\det R\left[h_0(t_0), \dots, h_{n-1}(t)\right] \neq 0, \qquad h_\nu(t_0) = \sum_{k=0}^{m} i^k a_\nu^{m-k,k}(t_0) \tag{3.1}$$

$$(\nu = 0, 1, \dots, n-1,\ t_0 \in \Gamma)$$

is necessary and sufficient for the validity of the assertions:

1a) the problem $A^0(\alpha_n)$ *has a finite number of linearly independent (over the field of real numbers) solutions; b) for the solvability of the problem* $A(\alpha_n)$ *it is necessary and sufficient that there be a finite number of conditions*

$$\int_\Gamma \psi_j(t) f(t)\, ds = 0, \tag{3.2}$$

where $\psi_j(t)$ $(j = 1, 2, \dots, k')$ *is a complete system of linearly independent solutions of the equation* $T'\psi = 0$, *the equation adjoint to* $T\mu = 0$.

2. The problem $A(\alpha_n)$ *is solvable up to finite-dimensional subspaces.*

If condition (3.1) is satisfied, by the **index** of the problem $A(\alpha_n)$ we shall mean the difference between the number of linearly independent solutions of the problem $A^0(\alpha_n)$ and the number of conditions (3.2) ensuring the solvability of the problem $A(\alpha_n)$.

Theorem 4. *The index of the problem $A(\alpha_n)$ is computed by the formula*

$$\varkappa(n) = 2[\varkappa^*(n) + mn], \qquad \varkappa^*(n) = \frac{1}{2\pi}\{\arg\det R\,[h_0(t), \dots, h_{n-1}(t)]\}_\Gamma, \quad (3.3)$$

where n is a natural number satisfying condition (1.1).

Corollary 1. *If condition (3.1) is satisfied and, in addition, $\varkappa^*(n) \geq 0$, then the problem $A^0(\alpha_n)$ is always solvable and has no fewer than $2[\varkappa^*(n) + mn]$ linearly independent solutions.*

Corollary 2. If condition (3.1) is satisfied and, moreover, $x^*(n) = -mn$, then problem $A(a_n)$ is always solvable and has a unique solution, provided that problem $A^0(a_n)$ has only the trivial solution. This solution is given by formula (1.5), where $\mu(t)$ is the solution of equation (1.7).

§ 4. If in the boundary conditions (1.3) we put $m = 1$, $b_\nu^{0,0} = b_\nu^{0,1} = b_\nu^{1,0} = 0$, we obtain the Poincaré problem with shift (problem $P(a_n)$), which, by introducing the notation

$$a^\nu(t) = a_\nu^{1,0}(t) + i a_\nu^{0,1}(t) \quad (\nu = 0, 1, \dots, n-1),$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \qquad 2\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y},$$

can be written as

$$\sum_{\nu=0}^{n-1}\left\{a^\nu(t_0)\frac{\partial u[\alpha_\nu(t_0)]}{\partial \alpha_\nu(t_0)} + \overline{a^\nu(t_0)}\frac{\partial u[\alpha_\nu(t_0)]}{\partial \overline{\alpha_\nu(t_0)}} + C_\nu(t_0)\,u[\alpha_\nu(t_0)]\right\} = f(t_0).$$

Condition (3.1) for problem $P(a_n)$ takes the form:

$$\lambda(t) = \det\{R[a_0^{1,0}(t), \dots, a_{n-1}^{1,0}(t)] + iR[a_0^{0,1}(t), \dots, a_{n-1}^{0,1}(t)]\} \neq 0 \quad (t \in \Gamma).$$

The index of this problem is computed by the formula

$$\varkappa(n) = 2[x^*(n) + n], \qquad x^*(n) = \frac{1}{2\pi}\{\arg\det\overline{\lambda(t)}\}_\Gamma.$$

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§ 5. The following special case of problem $A(a_n)$ is the Dirichlet problem with shift (problem $D(a_n)$). We obtain it if we put $m = 0$, $b_\nu^{0,0} = 0$ ($\nu = 0, 1, \dots, n-1$):

$$\sum_{\nu=0}^{n-1} a_\nu(t)\, u[\alpha_\nu(t)] = f(t). \tag{5.1}$$

This problem is also reduced to an equivalent singular integral equation with shift. The normality condition of the obtained equation consists in requiring that

$$\det R[a_0(t), \dots, a_{n-1}(t)] \neq 0 \quad (t \in \Gamma). \tag{5.2}$$

For this problem the theorems proved above are also valid in the corresponding formulations. The index of this problem is equal to zero.

If condition (5.2) is satisfied, problem $D(a_n)$ can be reduced directly to the ordinary Dirichlet problem for equation (1.2), which has been well studied (see ([1])).

In conclusion we note that the results of § 2 are valid when Γ consists of a finite number of simple closed contours. In view of this, using the method of I. N. Vekua (see ([1])), problem $A(a_n)$ is easy to study also for multiply connected domains.

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