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Abstract

Full Text

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QUASI-NILPOTENT GROUPS

(Presented by Academician A. I. Mal̆tsev on 30 XII 1963)

In the present paper one class of generalized nilpotent groups is introduced and studied; these groups are called quasi-nilpotent groups.

§ 1. A subgroup H of a group G is called **subinvariant** if there exists an ascending (generally speaking, transfinite) normal series in G having H as one of its terms. We shall call a group G **quasi-nilpotent** if every finite set of its elements generates a subinvariant subgroup of the group G.

From the example of an infinite locally nilpotent p-group without center, first constructed by A. G. Kurosh ([1]), and from a result of S. N. Chernikov ([2], theorem 13), it follows that the class of quasi-nilpotent groups is wider than the class of N-groups, i.e., groups with the normalizer condition.

It also follows immediately from this that the class of quasi-nilpotent groups is wider than the class of Baer nil-groups ([3]).

On the other hand, it follows from the definition that every quasi-nilpotent group belongs to the class of groups in which every cyclic subgroup is subinvariant. As is known ([4]), every group in which every cyclic subgroup is subinvariant is locally nilpotent. Consequently, every quasi-nilpotent group is a locally nilpotent group.

It is easy to prove that every countable locally nilpotent group is a quasi-nilpotent group.

From Theorem 7 of the present paper and from an example of M. I. Kargapolov ([5]) of a locally finite p-group having no ascending normal soluble series, it follows that **not every locally nilpotent group is a quasi-nilpotent group**.

§ 2. It is readily verified that every subgroup and every factor group of a quasi-nilpotent group are quasi-nilpotent groups. We note that the union of an ascending normal series of quasi-nilpotent groups will be a quasi-nilpotent group.

Let us record some properties of subinvariant subgroups.

1°. If a subgroup A of a group G is subinvariant in G, then it will be subinvariant in every subgroup of G in which it lies.

2°. The intersection of any two subinvariant subgroups of the group G will be subinvariant in G.

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3°. If a subgroup A is subinvariant in G, then the factor group AH/H will be subinvariant in the factor group G/H.

4°. If a subgroup A/H of the factor group G/H is subinvariant in G/H, then the subgroup A is subinvariant in G.

5°. If a subgroup A of the group G is subinvariant in a subgroup $B \subset G$ and the subgroup B is subinvariant in G, then the subgroup A will be subinvariant in G.

Lemma 1. *If in a locally nilpotent group G a subgroup H is subinvariant and the cyclic subgroup $\{g\}$ is subinvariant in G, where the element g belongs to the normalizer of the subgroup H in G, then the subgroup $\{H, g\} = H\{g\}$ will be subinvariant in G.*

Following ([4]), a group satisfying the maximality condition for subgroups will be called, briefly, an M-group. An M-group is also called a Noetherian group ([3]).

With the aid of the well-known fact (see also ([6]), corollary to Lemma 3) that the product of any finite number of normal nilpotent subgroups is itself nilpotent, the following lemma can be proved.

Lemma 2. If a group G can be represented as the product of two normal nilpotent M-subgroups A and B, i.e. $G = AB$, then the group G itself will be a nilpotent M-group, and in it one can construct a finite normal series of the form

$$A = A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots \subset A_n = G,$$

where

$$A_{i+1} = \{A_i, b_i\} = A_i\{b_i\}, \qquad [A_{i+1}, b_i] \subseteq A_i, \qquad b_i \in B.$$

Lemma 3. If a locally nilpotent group G is a cyclic extension of a ZA-group, then the group G itself is a ZA-group.

This lemma is proved with the aid of the well-known fact: in a nilpotent group every invariant subgroup and the center have a nontrivial intersection.

§ 3. With the aid of Lemmas 1 and 2 we obtain the following theorem.

Theorem 1. The product of any two invariant quasi-nilpotent subgroups of a group G will be an invariant quasi-nilpotent subgroup of the group G.

From this theorem the following result is obtained.

Theorem 2. The subgroup of any group G generated by all invariant quasi-nilpotent subgroups of the group G will be an invariant quasi-nilpotent subgroup of the group G.

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We shall denote this characteristic subgroup of the group G by $k(G)$ and call it the **quasi-nilpotent radical** of the group G. Obviously, if the group G itself is quasi-nilpotent, then $k(G) = G$.

From the examples noted above it follows that, in general, Baer' s nilradical $N(G)$ ([3,7]) is strictly contained in the quasi-nilpotent radical $k(G)$, while the quasi-nilpotent radical is strictly contained in the locally nilpotent radical $R(G)$ of B. I. Plotkin ([8]), i.e. $N(G) \subset k(G) \subset R(G)$.

§ 4. By analogy with ([9]), the following property of the quasi-nilpotent radical is proved.

Lemma 4. If H is an invariant subgroup of the group G, then $k(H)$ will be an invariant subgroup of the group G, and $k(H) = k(G) \cap H$.

Lemma 5. If a subgroup H is subinvariant in G, then $k(H)$ will be subinvariant in G, and $k(H) = k(G) \cap H$.

Corollary. If a group G has an ascending normal (invariant) series whose first term is a quasi-nilpotent subgroup of the group G, then the group G has a nontrivial quasi-nilpotent radical containing this first term.

Hence the following result is obtained:

Theorem 3. Every subinvariant nilpotent subgroup of a group G lies in $k(G)$, and, conversely, every subgroup generated by a finite number of generators and lying in $k(G)$ will be subinvariant in G.

Corollary. Every group G generated by subinvariant quasi-nilpotent subgroups is itself quasi-nilpotent.

This corollary proves once more that every N-group G is locally nilpotent (B. I. Plotkin' s theorem ([8])).

Theorem 4. A group G will be quasi-nilpotent if and only if all its cyclic subgroups are subinvariant.

Corollary. An M-group G is nilpotent if and only if all its cyclic subgroups are subinvariant (in particular, attainable) in G.

With the aid of Theorem 3 one proves

Theorem 5. For a subgroup A of any group G, the following properties are equivalent:

a) A is a subinvariant quasi-nilpotent M-subgroup;

b) A is generated by a finite number of subinvariant quasi-nilpotent M-subgroups;

c) A is an M-subgroup of the quasi-nilpotent radical of the group G.

This theorem is analogous to Baer' s theorem (([3]), Theorem 3, p. 418).

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§ 5. A group G possessing an ascending (in general, transfinite) normal series with abelian factors is called an RN^*-group. Since an extension of an RN^*-group by means of an RN^*-group is an RN^*-group, we obtain

Theorem 6. *In any group G, the subgroup generated by all normal RN^*-subgroups is a normal RN^*-subgroup of the group G.*

We shall call this characteristic subgroup of the group G the **RN^*-radical** of the group G and denote it by $RN^*(G)$.

For the radical $RN^*(G)$ the following holds.

Lemma 6. *If a subgroup H is subinvariant in G, then $RN^*(H)$ is subinvariant in G and*

$$RN^*(H) = RN^*(G) \cap H.$$

With the aid of this lemma one proves

Theorem 7*. *Every quasinilpotent group is an RN^*-group.*

From this theorem and from the indicated example of M. I. Kargapolov it follows that, in the general case, the radical $RN^*(G)$ is distinct from the locally nilpotent radical of B. I. Plotkin.

With the aid of Lemma 3 the following result is obtained:

Theorem 8. *Every locally nilpotent RN^*-group G is a quasinilpotent group.*

From Theorems 7 and 8 it follows

Theorem 9. *An RN^*-group G is quasinilpotent if and only if it is locally nilpotent.*

Corollary. *In any group G, the intersection of the locally nilpotent radical $R(G)$ of B. I. Plotkin and the radical $RN^*(G)$ coincides exactly with the quasinilpotent radical $k(G)$ of this group, i.e.*

$$k(G) = R(G) \cap RN^*(G).$$

§ 6. In the group G one can construct an ascending series of characteristic subgroups of the group G:

$$k(G) = k_1(G) \subset k_1(G) \subset k_2(G) \subset \dots \subset k_\alpha(G) \subset \dots$$

according to the following rules:

1) $k_{\alpha+1}(G)/k_\alpha(G) = k(G/k_\alpha(G))$;

2) if α is a limit ordinal, then

$$k_\alpha(G) = \bigcup_{\beta<\alpha} k_\beta(G).$$

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We shall call such a series a **quasinilpotent series** of the group G.

If γ is the first ordinal such that

$$k_{\gamma}(G) = k^{\gamma}_{\beta+1}(G),$$

then the (characteristic) subgroup $k_{\gamma}(G)$ will be called the **upper radical** of the group G and denoted by $\bar{k}(G)$. If $\bar{k}(G) = G$, then such a group will be called **quasinilpotently radical**.

From the fact that every RN^*-group possesses a nonunit quasinilpotent radical, the following characterization of RN^*-groups is obtained:

Theorem 10. *A group G is an RN^*-group if and only if it has an ascending series of characteristic subgroups with factors that are quasinilpotent groups.*

I consider it my duty to express my sincere gratitude to Prof. A. G. Kurosh and Prof. B. I. Plotkin for their attention and useful advice in carrying out the present work.

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* In connection with this theorem B. I. Plotkin proposed to the author the following question: will every locally nilpotent RN^*-group be a quasinilpotent group? Theorem 8 gives a positive answer to this question.

Note: Figure translations are in progress. See original paper for figures.

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