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Abstract

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MATHEMATICS

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ON THE EXISTENCE OF A SOLUTION OF A MULTI-INDEX LINEAR PROGRAMMING PROBLEM

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All unexplained notation is to be found in ([1]).
Consider the problem of minimizing the functional

$$L = \sum_{k \in M} p_{i_1 \dots i_s} x_{i_1 \dots i_s} \tag{1}$$

subject to the conditions

$$\sum_{k \in M_j} a^{(j)}_{i_1 \dots i_s} x_{i_1 \dots i_s} \begin{cases} \leqslant b_{M_j(i_1 \dots i_s)}, \\ = b_{M_j(i_1 \dots i_s)}; \end{cases} \tag{2}$$

$$x_{i_1 \dots i_s} \geqslant 0; \qquad j = 1, \dots, t; \qquad i_k = 1, \dots, n_k; \qquad k = 1, \dots, s. \tag{3}$$

All $a^{(j)}_{i_1 \dots i_s}$ and $p_{i_1 \dots i_s}$ are prescribed real numbers. In the particular case when all $a^{(j)}_{i_1 \dots i_s} = 1$, we obtain a multi-index linear programming problem of transportation type. In addition, all $b_{M_j(i_1 \dots i_s)} \geqslant 0$ are given.

We shall call the problem (1)–(3) an X-problem. The totality of all constraints for a fixed j will be called the j-block of constraints.

Let there exist $\widetilde{M} \neq \emptyset$ such that, if $\widetilde{M} \cap M_j \neq \widetilde{M}$, then all constraints entering the j-block of constraints are given in the form of inequalities. For definiteness, let $\widetilde{M} = \{r + 1, \dots, s\}$.

Consider

$$\prod_{k=r+1}^{s} n_k$$

independent problems:

$$\sum_{k\in M_j\setminus\widetilde{M}} a^{(j)}_{i_1\dots i_s} y_{i_1\dots i_r} \begin{cases} \leqslant b_{M_j(i_1\dots i_s)} \\ = b_{M_j(i_1\dots i_s)}. \end{cases} \qquad \text{for all } j \text{ such that } M_j \cap \widetilde{M} = \widetilde{M}; \quad (4)$$

$$\sum_{k\in M_j\setminus\widetilde{M}\cap M_j} a^{(j)}_{i_1\dots i_s} y_{i_1\dots i_r} \leqslant b_{M_j(i_1\dots i_s)} \quad \text{for all } j \text{ such that } M_j \cap \widetilde{M} \neq \widetilde{M}; \quad (5)$$

$$y_{i_1\dots i_r} \geqslant 0. \qquad (6)$$

In (4) and (5) all indices $i_{r+1}, \dots, i_s$ are fixed.

Theorem 1. *Suppose that for at least one set of values of the indices $i_{r+1} = \omega_{r+1}, \dots, i_s = \omega_s$ there exists a feasible solution of the problem (4)–(6). All $1 \leqslant \omega_k \leqslant n_k,\ k = r+1, \dots, s$. Then the X-problem also has a feasible solution.*

Proof. By direct substitution it is easy to verify that

$$x_{i_1\dots i_s} = \begin{cases} y_{i_1\dots i_r}, & \text{if } \bigcap\limits_{k\in\widetilde{M}} p(i_k = \omega_k) = 1, \\ 0, & \text{if } \bigcap\limits_{k\in\widetilde{M}} p(i_k = \omega_k) = 0, \end{cases}$$

is a feasible solution of the X-problem. Here $p(i_k = \omega_k)$ is a logical condition,

$$p(i_k = \omega_k) = \begin{cases} 1, & \text{if } i_k = \omega_k, \\ 0, & \text{if } i_k \neq \omega_k. \end{cases}$$

We shall say that the X-problem is **cut on the set of indices $\widetilde{M}$ with respect to the tuple** $(\omega_{r+1}, \dots, \omega_s)$.

Consider the $X(t)$-problem, which differs from the X-problem in that it does not include the t-block constraints.

Theorem 2. Suppose:

1) the $X(t)$-problem is cut on the set of indices

$$\widetilde{M} \equiv \bigcup_{j=1}^{t-1} M_j \setminus M_t$$

for all tuples $(\omega_{r+1}, \dots, \omega_s)$;

2) there exists at least one $j_* \neq t$ such that all $a^{(j_*)}_{i_1 \dots i_s} = a^{(t)}_{i_1 \dots i_s}$;

3) the j_*-block constraints and the t-block constraints are given in the form of equalities;

4) the compatibility conditions are satisfied,

$$\sum_{k \in M_{j_*} \cup M_t \setminus M_{j_*}} b_{M_{j_*}(i_1 \dots i_s)} = \sum_{k \in M_{j_*} \cup M_t \setminus M_t} b_{M_t(i_1 \dots i_s)}.$$

Then the X-problem has a feasible solution.

Proof. Every solution of the $X(t)$-problem can be written in the form

$$x^{(\omega_{r+1} \dots \omega_s)}_{i_1 \dots i_s} = \begin{cases} y_{i_1 \dots i_r}, & \text{if } \bigcap_{k \in \widetilde{M}} p(i_k = \omega_k) = 1, \\ 0, & \text{if } \bigcap_{k \in \widetilde{M}} p(i_k = \omega_k) = 0. \end{cases}$$

We shall show that there exist such

$$\lambda_{\omega_{r+1} \dots \omega_s} \geq 0, \qquad \sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} \lambda_{\omega_{r+1} \dots \omega_s} = 1, \tag{7}$$

for which the convex combination

$$\sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} x^{(\omega_{r+1} \dots \omega_s)}_{i_1 \dots i_s} \lambda_{\omega_{r+1} \dots \omega_s} = \bar{x}_{i_1 \dots i_s}$$

satisfies the t-block constraints

$$\sum_{k \in M_t} a^{(t)}_{i_1 \dots i_s} \bar{x}_{i_1 \dots i_s} = \sum_{k \in M_t} a^{(t)}_{i_1 \dots i_s} \left(\sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} x^{(\omega_{r+1} \dots \omega_s)}_{i_1 \dots i_s} \lambda_{\omega_{r+1} \dots \omega_s} \right) =$$

$$= \sum_{k \in M_t} a^{(t)}_{i_1 \dots i_s} x^{(i_{r+1} \dots i_s)}_{i_1 \dots i_s} \lambda_{i_{r+1} \dots i_s} = \sum_{k \in M_t} a^{(t)}_{i_1 \dots i_s} y_{i_1 \dots i_r} \lambda_{i_{r+1} \dots i_s} = b_{M_t(i_1 \dots i_s)}.$$

Since, by definition, $\widetilde{M} \cap M_t = \emptyset$, it follows that

$$\lambda_{i_{r+1}\dots i_s} = \frac{b_{M_t(i_1\dots i_s)}}{\sum\limits_{k\in M_t} a^{(t)}_{i_1\dots i_s} y_{i_1\dots i_r}}.$$

It remains to show that condition (7) is satisfied. From the compatibility conditions,

$$\sum_{k\in M_{j_*}\cup M_t\setminus M_t} b_{M_t(i_1\dots i_s)} = \sum_{k\in M_{j_*}\cup M_t} \left(\sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} a^{(t)}_{i_1\dots i_s} x^{(\omega_{r+1}\dots\omega_s)}_{i_1\dots i_s} \lambda_{\omega_{r+1}\dots\omega_s} \right) =$$

$$= \left(\sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} \lambda_{\omega_{r+1}\dots\omega_s} \right) \left(\sum_{k\in M_{j_*}\cup M_t} a^{(j_*)}_{i_1\dots i_s} x^{(\omega_{r+1}\dots\omega_s)}_{i_1\dots i_s} \right) =$$

$$= \left(\sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} \lambda_{\omega_{r+1}\dots\omega_s} \right) \left(\sum_{k\in M_{j_*}\cup M_t\setminus M_{j_*}} b_{M_{j_*}(i_1\dots i_s)} \right)$$

it follows that

$$\sum_{\omega_{r+1}=1}^{n_{r+1}} \cdots \sum_{\omega_s=1}^{n_s} \lambda_{\omega_{r+1}\dots\omega_s} = 1.$$

The theorem is proved.

Consider a multi-index linear programming problem of transportation type, in which all constraints are given in the form of equalities, i.e., conditions (2) are written as follows:

$$\sum_{k\in M_j} x_{i_1\dots i_s} = b_{M_j(i_1\dots i_s)}. \qquad (2\ )$$

Denote $J_0 \equiv \{1, \dots, t\}$, $M_j^{(0)} \equiv M_j$. For each $r = 1, \dots, t-2$, consider

$$M_j^{(r)} \equiv M_j^{(r-1)} \setminus \bigcap_{j\in J_{r-1}} M_j^{(r-1)} \quad \text{for all } j \in J_{r-1},$$

$$M^{(r)} \equiv \bigcup_{j\in J_{r-1}} M_j^{(r-1)}.$$

Theorem 3. If for each $r = 1, \dots, t-2$ there exists, in the last resort, one $j_r \in J_{r-1}$ such that

$$M^{(r)} \setminus M^{(r)}_{j_r} \subset \bigcap_{j \in J_r} M^{(r)}_j, \quad \text{where } J_r = J_{r-1} \setminus j_r,$$

then the multi-index problem of transportation type has a feasible solution if and only if the consistency conditions are satisfied.

Sufficiency. For $t = 2$, the consistency conditions are sufficient for any M_1 and M_2. In this case the feasible solution is written as follows:

$$x_{i_1 \dots i_s} = \frac{b_{M_1(i_1 \dots i_s)}\, b_{M_2(i_1 \dots i_s)}}{\sum\limits_{k \in M_2^{(1)}} b_{M_1(i_1 \dots i_s)}}.$$

Consider the case $t = 3$. There exists j_1 such that

$$M^{(1)} \setminus M^{(1)}_{j_1} \subset \bigcap_{j \in J_1} M^{(1)}_j. \tag{8}$$

It is not hard to see that J_r contains $t - r$ elements, i.e., for $t = 3$, J_1 contains two elements.

Since for the $X(j_1)$-problem all the conditions of Theorem 2 are satisfied, for $t = 3$ the problem (1), (2), (3) has a feasible solution.

Suppose that the theorem is true for $t = \tau$. We shall show that it is true for $t = \tau + 1$. In this case condition (8) is also satisfied, and J_1 contains τ elements. Since the $X(j_1)$-problem has a solution, the X-problem also has a solution by Theorem 2. Necessary and sufficient conditions for the existence of a feasible solution to problem (2), (3) were obtained in [2] by another method, in considering mixed strategies in coalition games.

Consider the case when all $a^{(j)}_{i_1 \dots i_s} \geq 0$ and upper and lower bounds are imposed on all $x_{i_1 \dots i_s}$, i.e.,

$$0 \leq m^{(1)}_{i_1 \dots i_s} \leq x_{i_1 \dots i_s} \leq m^{(0)}_{i_1 \dots i_s} \leq \infty. \tag{3 }$$

Denote

$$\sum_{k \in M_j} m^{(l)}_{i_1 \dots i_s} x_{i_1 \dots i_s} \equiv A^{(l,l)}_{M_j(i_1 \dots i_s)}.$$

We shall compute

$$m^{(l)}_{i_1\dots i_s} = (-1)^l \min_{1\le j\le t}\left[(-1)^l\left(\frac{b_{M_j(i_1\dots i_s)} - A^{(j,l-1)}_{M_j(i_1\dots i_s)}}{a^{(j)}_{i_1\dots i_s}} + m^{(l-1)}_{i_1\dots i_s}\right), (-1)^l m^{(l-2)}_{i_1\dots i_s}\right].$$

Theorem 4. In order that problem (2), (3) have a solution, it is necessary that the condition

$$(-1)^l\left[m^{(l)}_{i_1\dots i_s} - m^{(l-1)}_{i_1\dots i_s}\right] \ge 0$$

be satisfied for all $l = 2, 3, \dots;\ i_k = 1, \dots, n_k;\ k = 1, \dots, s$. Moreover, after a finite number of steps l, either all

$$m^{(l)}_{i_1\dots i_s} = m^{(l-2)}_{i_1\dots i_s},$$

and in this case all lower and upper bounds are balanced, or at least in one cell the condition

$$(-1)^l\left[m^{(l)}_{i_1\dots i_s} - m^{(l-1)}_{i_1\dots i_s}\right] \ge 0$$

is violated, i.e. problem (2), (3) has no feasible solution.

Consider the multi-index LP problem of transportation type.

Theorem 5. If $\mu_{i_1\dots i_s}$ and $M_{i_1\dots i_s}$ are, respectively, the values of the balanced lower and upper bounds, then the inequalities

$$\sum_{k\in M_j} M_{i_1\dots i_s} + \left(\prod_{l\in M_j} n_l - 1\right)\sum_{k\in M_j}\mu_{i_1\dots i_s} \le \left(\prod_{l\in M_j} n_l\right) b_{M_j(i_1\dots i_s)} \le$$

$$\le \sum_{k\in M_j}\mu_{i_1\dots i_s} + \left(\prod_{l\in M_j} n_l - 1\right)\sum_{k\in M_j} M_{i_1\dots i_s} \qquad \text{for all } j = 1, \dots, t.$$

Special cases of Theorems 4 and 5 were considered by the author in work ([3]).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.