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Abstract

Full Text

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A CLOSED-GRAPH THEOREM FOR ULTRA-COMPLETE SPACES

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The paper proves a closed-graph theorem for multivalued mappings of ultracomplete spaces. Theorems of this kind for linear mappings of locally convex spaces were established by V. Pták ([1]) and J. L. Kelley ([2]). V. L. Levin and D. A. Raikov ([3]) proved analogous theorems in the case of so-called biregular mappings, for one class of uniform spaces (including ultracomplete spaces) which they called B-complete.

Let us introduce some notation that will be useful below. Let X and Y be uniform spaces (not assumed to be separated). We shall regard any subset of the product $X \times Y$ as the graph of a multivalued mapping from X into Y, and denote it by the same letter as this mapping. In particular, entourages in X are regarded as multivalued mappings of the space X into itself.

By $\{U_i\}_{i\in I}$ we denote an arbitrary fundamental system of symmetric entourages in X, and by $\{\tilde{U}_i\}_{i\in I}$ a fundamental system of symmetric entourages in Y. It is clear that the set of indices in both cases may be taken to be one and the same. With each entourage V we shall associate a sequence of entourages $V^{(1)}, \dots, V^{(n)}, \dots$, satisfying the conditions

$$V^{(1)}V^{(1)} \subset V; \qquad V^{(n+1)}V^{(n+1)} \subset V^{(n)} \quad (n = 1, 2, \dots).$$

Following ([3]), a multivalued mapping F of the space X into the space Y will be called uniformly almost open (respectively, uniformly open) if for every entourage U in X there exists an entourage U^F in Y such that, for every point $x_0 \in X$, the inclusion

$$\overline{FU}(x_0) \supset U^F F(x_0)$$

holds (respectively,

$$FU(x_0) \supset U^F F(x_0)).$$

The Hausdorff uniformity on the set $\mathfrak{F}(X)$ of all closed subsets of the space X is the uniformity defined by the base of entourages

$$\{(A, B) : A \subset U(B),\ B \subset U(A)\},$$

where U runs through any fundamental system of entourages of the uniformity given on X. If $\mathfrak{F}(X)$, endowed with the Hausdorff uniformity, is complete, then the space X is called ultracomplete.

Lemma*. Let F be a uniformly almost open multivalued mapping with closed graph from an ultracomplete space X into an arbitrary uniform space Y. Then, for any entourages U and V in X and any point $x_0 \in X$, the inclusion

$$FVU(x_0) \supset \overline{FU}(x_0)$$

is valid.

Proof. Obviously, one may assume that

$$U_i^{(n)} \subset V^{(n+1)}$$

for all i and all n.

Let

$$\bar{y} \in \overline{FU}(x_0).$$

We shall construct a fundamental (in the sense of the Hausdorff uniformity) generalized sequence of sets

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* In the special case when X is a complete pseudometric space, this assertion was proved by J. Kelley (([4]), pp. 202–203).

$C_\mu \subset X$, such that: 1) all C_μ lie in $V^{(1)}U(x_0)$, so that $C = \lim_\mu \overline{C}_\mu$ lies in $VU(x_0)$; 2) $\bar{y} \in F(\bar{x})$ for any point $\bar{x} \in C$; 3) the index μ runs through all possible finite collections of elements of the set I (ordered by inclusion).

First we construct an auxiliary sequence of sets B_μ satisfying the conditions: a) all B_μ lie in $V^{(1)}U(x_0)$; b) if $\mu \supset \nu$, then

$$B_\nu \subset \bigcap_{i\in\nu} U_i^{(|\nu|)}(B_\mu),$$

where $|\nu|$ is the number of elements of the set ν; c) for any point $x \in B_\mu$ the relation

$$\bigcap_{i\in\mu} \widetilde{U}_i(\bar{y}) \cap F(x) \neq \emptyset$$

holds.

If such sets B_μ have been constructed, then it suffices to put

$$C_\mu = \bigcup_{\nu\supset\mu} B_\nu.$$

Indeed, let us verify the fundamentality of the system $\{C_\mu\}$. If $\mu \supset \nu$, then $C_\mu \subset C_\nu$. Therefore, for each neighborhood W it is necessary to find a μ_0

such that from $\mu \supset \mu_0$ it follows that $W(C_\mu) \supset C_{\mu_0}$. As such a μ_0 take a set consisting of one index i satisfying $U_i \subset W$. Let $x \in C_i$; then $x \in B_\nu$, where $i \in \nu$, and let $\xi = \mu \cup \nu$. Then $B_\xi \subset C_\mu$, and

$$x \in B_\nu \subset \bigcap_{j\in\nu} U_j^{(|\nu|)}(B_\xi) \subset U_i(B_\xi) \subset W(B_\xi),$$

as was required. Thus, the sequence C_μ is fundamental and satisfies conditions 1) and 3). Let us verify condition 2).

If $\bar{x} \in C$, then every neighborhood of the point $\bar{x}$ contains points from C_μ for all sufficiently large μ, and hence also points from B_ν for ν belonging to some cofinal subsequence of the set of indices. From this and from property c) of the sets B_ν it follows that for given neighborhoods U and $\widetilde{U}$ of the points $\bar{x}$ and $\bar{y}$, respectively, there exist points $x \in U(\bar{x})$ and $y \in \widetilde{U}(\bar{y})$ for which $y \in F(x)$, i.e. $(x, y) \in F$. From the closedness of the graph it follows that $\bar{y} \in F(\bar{x})$.

It remains to construct the sets B_μ. We shall construct them by induction on $|\mu|$, the number of elements in μ.

$$B_i = \{x : \ x \in U(x_0), \ \widetilde{U}_i(\bar{y}) \cap U_i^F(\bar{y}) \cap F(x) \neq \emptyset\};$$

$$B_\mu = \{x : \ x \in \bigcup_{\substack{\nu\subset\mu \\ \nu\neq\mu}} \bigcap_{i\in\nu} U_i^{(|\nu|)}(B_\nu), \ \left(\bigcap_{i\in\mu} \widetilde{U}_i(\bar{y})\right) \cap \left(\bigcap_{i\in\mu} U_i^{(|\mu|)}\right)^F (\bar{y}) \cap F(x) \neq \emptyset\}.$$

Since

$$\bar{y} \in \overline{FU(x_0)},$$

the intersection of any neighborhood of the point $\bar{y}$ with the set $FU(x_0)$ is nonempty. Hence the nonemptiness of the sets B_i follows. At the same time it is easy to see that $\bar{y} \in \overline{F(B_i)}$, and, consequently, the same reasoning proves the nonemptiness of the sets B_μ. Let us prove that requirements a), b), c) are satisfied.

For all B_i the inclusion $B_i \subset U(x_0)$ is satisfied. Suppose that for all ν for which $|\nu| < k$, the inclusion

$$B_\nu \subset V^{(|\nu|)} V^{(|\nu|-1)} \dots V^{(2)} U(x_0)$$

is satisfied, and let $|\mu| = k$. Then

$$B_\mu \subset \bigcup_{\substack{\nu\subset\mu\\ \nu\neq\mu}} \bigcap_{i\in\nu} U_i^{(|\nu|)}(B_\nu) \subset \bigcup_{\substack{\nu\subset\mu\\ \nu\neq\mu}} V^{(|\nu|+1)}(B_\nu) \subset$$

$$\subset \bigcup_{\substack{\nu\subset\mu\\ \nu\neq\mu}} V^{(|\nu|+1)} V^{(|\nu|)} \dots V^{(2)} U(x_0) = V^{(|\mu|)} \dots V^{(2)} U(x_0).$$

It follows that all B_μ are contained in $V^{(1)}U(x_0)$, i.e. assertion a).

Now let $x \in B_\nu$ and $\mu \supset \nu$. Then from the relation

$$\left(\bigcap_{i\in\nu} U_i^{(|\nu|)}\right)^F (\bar{y}) \cap F(x) \neq \emptyset,$$

i.e.

$$\bar{y} \in \left(\bigcap_{i\in\nu} U_i^{(|\nu|)}\right)^F F(x) \subset \overline{F\left(\bigcap_{i\in\nu} U_i^{(|\nu|)}\right)(x)}$$

there follows the exist–

of the existence of a point $x' \in \bigcap_{i\in\nu} U_i^{(|\nu|)}(x)$ such that

$$\left(\bigcap_{i\in\mu} \overline{U_i}(\bar{y})\right) \cap \left(\bigcap_{i\in\mu} U_i^{(|\mu|)}\right)^F (\bar{y}) \cap F(x') \neq \emptyset,$$

and this means that $x' \in B_\mu$ (recall that $x \in B_\nu$). Using the symmetry of the neighborhoods, we obtain $x \in \bigcap_{i\in\nu} U_i^{(|\nu|)}(x')$, i.e. $B_\nu \subset \bigcap_{i\in\nu} U_i^{(|\nu|)}(B_\mu)$. Thus condition b) is fulfilled. Property c) follows from the definition of the sets B_μ.

The lemma is proved. It immediately implies the following

Theorem. *If the space X is ultracomplete, then every uniformly almost open multivalued mapping with closed graph from the space X into an arbitrary uniform space Y is uniformly open.*

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.