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Abstract

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ASYMPTOTIC METHOD FOR SOLVING A FOURTH-ORDER DIFFERENTIAL EQUATION WITH TWO SMALL PARAMETERS IN THE HYDRODYNAMIC THEORY OF STABILITY

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In the problem of natural oscillations in media with slowly varying parameters, the method of geometrical optics (the WKB approximation), well developed for second-order differential equations, is often applied. Often, however, in various applications the problem of natural oscillations leads to the need to study differential equations of higher order. A classical example is the Orr–Sommerfeld equation, well known in the theory of hydrodynamic stability ([1]), containing a small parameter at the fourth derivative. We shall be interested, in the case of a weakly inhomogeneous medium, in the differential equation

$$\alpha\beta^2\varphi^{\mathrm{IV}} - \beta U_2(x,k,\omega)\varphi'' + U_1(x,k,\omega)\varphi = 0, \tag{1}$$

where $k,\ \omega$ are, respectively, the wave vector and the frequency of the wave; β is a small parameter of "quasiclassicality," taking into account the weak inhomogeneity in x; α is a small parameter associated with the particular formulation of the problem (in the Orr–Sommerfeld equation, for example, α is proportional to the viscosity); $U_1, U_2 \sim 1$ except for small regions near points where U_1 and U_2 vanish. We note that, as will be seen below, the WKB method breaks down not only near points where $U_1 = 0$ (near such points, for two of the solutions of (1), the wave vector becomes small), but also near points where $U_2 = 0$. In this connection there arises the problem, taking into account the indicated features, of obtaining rules for finding the eigenfrequencies ("quantization rules") for finite solutions of (1).

For convenience we choose a specific form of U_1, U_2, as shown in Fig. 1. Far from A, B, O_1, O_2 (turning points), solutions of (1) are sought in the form of an asymptotic series in the small parameter $\sqrt{\beta}$ and have the form:

$$\varphi_{1,2} = \frac{\text{const}}{\sqrt{q_{1,2}}} \exp \frac{1}{\sqrt{\beta}} \int^x q_{1,2}(x)\, dx, \tag{2}$$

$$\varphi_{3,4} = \frac{\text{const}}{\sqrt{q_{3,4}^5}} \exp \frac{1}{\sqrt{\beta}} \int^x q_{3,4}(x)\, dx, \tag{3}$$

where*

$$q_{1,2} = \pm\sqrt{\frac{U_2}{2\alpha} - \sqrt{\frac{U_2^2}{4\alpha^2} - \frac{U_1}{\alpha}}}, \quad q_{3,4} = \pm\sqrt{\frac{U_2}{2\alpha} + \sqrt{\frac{U_2^2}{4\alpha^2} - \frac{U_1}{\alpha}}}. \tag{4}$$

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* Here and below, for convenience of notation, the pre-exponential factor valid for $x > \sqrt{\alpha}$ is written out.

To obtain the solution near the points where $U_2 = 0$, we set $U_2 = Ux$ $(U \sim 1)$, $x = \beta y$. This leads to the equation

$$\frac{\alpha}{\beta^2}\varphi^{\text{IV}} - U_y\varphi'' + U_1\varphi = 0. \tag{5}$$

It is not difficult to see that near the point where $U_2 = 0$, there exist two points at which, respectively, $q_1 = q_3$, $q_2 = q_4$ (points of "intersection" of the solutions). Near the indicated points, a strict separation of the "normal" solutions (2) and (3) from one another is, generally speaking, impossible—they "transform" into one another.

The physical picture of the solution of the problem depends essentially on the magnitude of the parameter α/β^2. If $\alpha/\beta^2 \ll 1$, then, as is seen from (4), the distance between the points of "intersection" of the solutions is small compared with the wavelength of the intersecting solutions. We note that only this case has been investigated in the study of Poiseuille flow ([1]), and also in connection with other physical problems ([2]). In this case the connection between the solutions $\varphi_{1,2}$ and $\varphi_{3,4}$ is effected only in the next order in α/β^2* (weak coupling). In the case under consideration the finite solutions correspond to the following "quantization rules" ([4]):

$$\int_{O_1}^{O_2} \sqrt{\frac{U_2}{\alpha\beta}}\, dx = \left(n + \frac{1}{2}\right)\pi; \qquad \int_{O_2}^{A} \sqrt{\frac{U_1}{\beta U_2}}\, dx = \left(n + \frac{1}{2}\right)\pi. \tag{6}$$

A qualitatively new picture arises when $\alpha/\beta^2 \gg 1$. In this case, around each point of "intersection" of solutions, one can always distinguish such a region in

Fig. 1 and Fig. 2 diagrams

Figure 1: Fig. 1 and Fig. 2 diagrams

the complex x-plane where the "quasiclassical" approximation (2), (3) is valid for the solution of (5). Obviously, near such points a solution of type (2) can already in the zeroth approximation "transform" into a solution of type (3), and conversely (strong coupling).

Fig. 1 **Fig. 2**

We obtain the solutions of equation (5) by applying the Laplace method:

$$\varphi(y) = \int \frac{1}{t^2} \exp\left\{yt - \frac{\alpha}{\beta^2}\frac{t^3}{3U} + \frac{1}{t}\frac{U_1}{U}\right\} dt, \tag{7}$$

where the integral is taken in the plane of the complex variable t along a contour at whose ends the function

$$\exp\left\{yt - \frac{\alpha}{\beta^2}\frac{t^3}{3U} + \frac{1}{t}\frac{U_1}{U}\right\}$$

vanishes.

The solution (7), like equation (5) itself, is valid in the region $y < 1/\beta$. For

$$1 < y < 1/\beta. \tag{8}$$

to compute (7) one may use the saddle-point method and obtain the following four linearly independent solutions:

$$\varphi_i(y) \sim \sqrt{\frac{\pi}{y\left(\frac{U_1}{U}\frac{1}{q_i^3} - \frac{\alpha}{\beta^2}\frac{\bar{q}_i}{U}\right)}}\frac{1}{q_i^2}\exp\int^y q_i(y)\,dy \qquad (i = 1, 2, 3, 4), \tag{9}$$

* With the exception of a certain region of the complex x-plane (see (3)).

where

$$\begin{aligned} \bar{q}_{1,2} &= \pm\sqrt{\frac{\beta^2}{\alpha}\frac{U}{2}}\sqrt{y - \sqrt{y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}}}, \\ \bar{q}_{3,4} &= \pm\sqrt{\frac{\beta^2}{\alpha}\frac{U}{2}}\sqrt{y + \sqrt{y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}}}. \end{aligned} \tag{10}$$

Using (10), we obtain the solutions (9) in the form

$$\varphi_{1,2} \sim (\bar{q}_i)^{-1/4} \exp \int^y \bar{q}_i(y)\, dy, \qquad \varphi_{3,4} \sim (\bar{q}_i)^{-5/4} \exp \int^y \bar{q}_i(y)\, dy, \tag{11}$$

which respectively pass into (2) and (3). The "intersection"points of the solutions correspond to

$$y_0 \equiv ia = \pm\sqrt{\frac{\alpha}{\beta^2}\frac{4U_1}{U^2}}. \tag{12}$$

It follows from (12) and (8) that, when $\alpha/\beta^2 \gg 1$, in accordance with what was said above, the points y_0 can always be surrounded by a region where one may use the solution (9), which passes into the "quasiclassical" solutions (2) and (3).

We note that in the immediate vicinity of the points y_0, the contours in the t-plane corresponding to the linearly independent solutions (5) coalesce, and an additional study of the character of the solutions at the points y_0 is required. We shall not, however, be interested in this question, since for obtaining the quasiclassical "quantization rules" it is sufficient to know the rules for going around the points y_0 in the complex x-plane.

We use the representation:

$$\pm\sqrt{y-\sqrt{y^2+a^2}} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{y+ia}-\sqrt{y-ia}\right) \qquad (y>0),$$

$$= \pm\frac{i}{\sqrt{2}}\left(\sqrt{|y|+ia}-\sqrt{|y|-ia}\right) \qquad (y<0);$$

$$\pm\sqrt{y+\sqrt{y^2+a^2}} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{y+ia}+\sqrt{y-ia}\right) \qquad (y>0),$$

$$= \pm\frac{i}{\sqrt{2}}\left(\sqrt{|y|+ia}+\sqrt{|y|-ia}\right) \qquad (y<0).$$

After this, the solutions (2), (3) for (5) are written in the form:

$$\varphi_{1,2} = (w_1-w_2)^{-1/2} \exp\left\{\pm i\int^y (w_1(y)-w_2(y))\, dy\right\} \qquad (y>0),$$

$$= (w_1-w_2)^{-1/2} \exp\left\{\pm\int^y (w_1(|y|)-w_2(|y|))\, dy\right\} \qquad (y<0); \tag{13}$$

$$\varphi_{3,4} = (w_1 + w_2)^{-5/2} \exp\left\{\pm \int^y (w_1(y) + w_2(y))\, dy\right\} \qquad (y > 0),$$

$$= (w_1 + w_2)^{-1/2} \exp\left\{\pm i \int^y (w_1(|y|) + w_2(|y|))\, dy\right\} \qquad (y < 0),$$

where

$$w_1 = \sqrt{\frac{\beta^2 U}{2\alpha}(y - ia)}, \qquad w_2 = \sqrt{\frac{\beta^2 U}{2\alpha}(y + ia)}.$$

Using formulas (13), one can construct the picture of the level lines of w_1, w_2 for each of the intersection points of the solutions separately (Fig. 2)*. The rules for stitching the solutions (13) near the point O_1 are obtained in the following way. From (13) it is seen that one can go around separately in the complex y-plane about the points $a_1 = ia$ and $a_2 = -ia$. When going around a_1, the pairs of solutions (φ_1, φ_4) and (φ_2, φ_3), and when going around a_2, the pairs (φ_1, φ_3) and (φ_2, φ_4), behave independently. Denote by A_i, B_i, C_i, D_i the systems of coefficients of the solution for $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, respectively, near the lines numbered i, issuing from

* The meaning of all notations and letters not specified in the text is clear from Fig. 2.

points a_1, a_2. Using, in going around each point separately, rules of the type (5), after simultaneously going around a_1, a_2 we obtain:

$$\begin{gathered}
A_2 = A_1 + iD_1, \quad B_2 = B_1 + iD_1, \quad C_2 = iA_1 + iB_1 + C_1 - D_1, \quad D_2 = D_1, \\
A_3 = A_2 + iC_2, \quad B_3 = B_2 + iC_2, \quad C_3 = C_2, \quad D_3 = iA_2 + iB_2 - C_2 + D_2, \\
A_1' = A_3 + iD_3, \quad B_1' = B_3 + iD_3, \quad C_1' = iA_3 + iB_3 + C_3 - D_3, \quad D_1' = D_3, \\
A_1' = -B_1, \quad B_1' = -A_1, \quad C_1' = -D_1, \quad D_1' = -C_1.
\end{gathered} \tag{14}$$

To the left of the point A we write an arbitrary solution tending to zero as $-\infty$:

$$\varphi = |w_1 - w_2|^{-1/2} \exp\left\{-i \int_A^y w_1(y)\, dy + i \int_A^y w_2(y)\, dy\right\} +$$

$$+ D\, |w_1 + w_2|^{-5/2} \exp\left\{\int^y w_1(y)\, dy + \int^y w_2(y)\, dy\right\}. \tag{15}$$

Using (13)–(15) and requiring finiteness of the solution as $+\infty$, we obtain the following "quantization rules" :

$$\Phi_1 + \Phi_2 + \Phi_3 = \left(n + \frac{1}{2}\right)\pi, \qquad \Phi_1 = i\int_{L_1} p_1\,dz - i\int_{L_2} p_2\,dz,$$

$$\Phi_2 = \int_{L_3} p_1\,dz + \int_{L_4} p_2\,dz, \qquad \Phi_3 = -i\int_{L_1'} p_1\,dz + i\int_{L_2'} p_2\,dz, \tag{16}$$

$$p_1 = \sqrt{\frac{\beta^2}{2\alpha}\left(U_2 - \sqrt{4U_1\alpha/\beta^2}\right)}, \qquad p_2 = \sqrt{\frac{\beta^2}{2\alpha}\left(U_2 + \sqrt{4U_1\alpha/\beta^2}\right)}.$$

Here the contours L_1, L_2, beginning at the point A, go along the real axis and end: the contour L_1 at the point a_1, and L_2 at the point a_2. The contours L_3, L_4 begin respectively at the points a_1, a_2, then descend to the real axis, go along it, and end respectively at the points b_1, b_2. The contours L_1', L_2' are analogous to the contours L_1, L_2, with a_1, a_2 replaced by b_1, b_2 and A by B. The expressions for p_1, p_2 in the vicinity of the points O_1, O_2 pass respectively into w_1, w_2.

It is not difficult to verify that the quantities Φ_1, Φ_2, Φ_3 defined in (18) are purely real. Consider, for example, Φ_2. Taking into account that on the real axis L_3 and L_4 coincide, we have: $\int_{O_1}^{O_2}(p_1 + p_2)\,dy$ is purely real. Moreover,

$$\int_{a_2}^{O_1} \sqrt{z + ia}\,dz = (ia)^{3/2}, \qquad \int_{a_1}^{O_1} \sqrt{z - ia}\,dz = (-ia)^{3/2}.$$

From this the assertion made immediately follows.

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Note: Figure translations are in progress. See original paper for figures.