A. F. LAVRIK

1964

SovietRxiv

Abstract

Full Text

A. F. LAVRIK

SUM OVER CHARACTERS OF POWERS OF THE MODULUS OF DIRICHLET L-FUNCTIONS IN THE CRITICAL STRIP

(Presented by Academician I. M. Vinogradov on 16 VII 1963)

The problem of the distribution of the numbers $m_1 \dots m_k$ in short intervals of arithmetic progressions, which has a number of important applications, is closely connected with estimates of sums over characters of powers of L-functions in the critical strip, i.e., sums of the form

$$\sum_{\chi \bmod D} |L(s,\chi)|^k; \qquad s = \sigma + it, \quad \frac{1}{2} \le \sigma < 1. \tag{1}$$

Yu. V. Linnik in ([1]) obtained the necessary estimates of the sums (1) for $\sigma = \frac{1}{2}$, $k \le 4$, which enabled him to indicate important (see ([2])) asymptotic laws of distribution of the numbers $m_1 \dots m_k \le N$ for $k \le 4$ in progressions $Dn + l$ with D, close to $\sqrt{N}$. At the same time, as is clear from his work ([3]), the extension of these results for $\sigma = \frac{1}{2}$ to the case $k \ge 5$ encounters difficulties that are as yet insurmountable.

However, the result of Yu. V. Linnik admits the possibility of substantial generalizations in another direction. Namely, by shifting σ to 1 one succeeds in obtaining the needed estimate of the sums (1) for arbitrary $k \ge 2$. This, in turn, makes it possible to obtain asymptotic laws for the number of numbers $m_1 \dots m_k \le N$ belonging to the progression $Dn + l$ with difference $D \le N^\alpha$, where $\alpha = \alpha(k)$ is a constant for any integer k.

Theorem 1. *Let D be an integer > 1; χ a character modulo D; $L(s,\chi)$ the corresponding Dirichlet L-series. Then for every integer $k \ge 1$, uniformly in D and t,*

$$\sum_{\chi \bmod D} \left| L\left(1 - \frac{1}{k} + it, \chi\right) \right|^{2k} = B\varphi(D)(|t|+1) \ln^b D(|t|+1), \tag{2}$$

where the sum extends over all $\varphi(D)$ characters modulo D; b is a constant depending only on k.

With the aid of Theorem 1, the following results are obtained concerning the distribution of the numbers $m_1 \dots m_k$ in progressions.

Theorem 2. *Let $\tau_{2k}(n)$ denote the number of solutions in integers $m_i \geq 1$ of the equation $n = m_1 \ldots m_{2k}$; $(l, D) = 1$; $\gamma < \frac{1}{k}$; δ is a constant > 1;*

$$D \leq x^{\gamma} \exp\left(-\ln^{4\delta} \ln x\right). \tag{3}$$

Then, for $x > x_0$, for every integer $k \geq 1$,

$$\sum_{\substack{n \leq x \\ n \equiv l \pmod D}} \tau_{2k}(n) = \frac{1}{\varphi(D)} \operatorname*{res}_{s=1} \left\{\frac{x^s}{s} L^{2k}(s, \chi_0)\right\} + R(x; D),$$

where φ is Euler' s function; χ_0 is the principal character modulo D;

$$R(x; D) = O\left(\frac{x^{1-\frac{1-k\gamma}{4k}}}{\varphi(D)} e^{-\frac{1}{2}\ln^{\delta}\ln x}\right).$$

Under the conditions of Theorem 2 there holds

Theorem 3.

$$\sum_{\substack{n \leq x \\ n \equiv l \pmod D}} \tau_{2k}(n) =$$

$$= \frac{1}{\varphi(D)} \sum_{d_1 | D} \cdots \sum_{d_{2k} | D} \mu(d_1) \cdots \mu(d_{2k}) \sum_{n \leq \frac{x}{d_1 \cdots d_{2k}}} \tau_{2k}(n) + R(x; D), \tag{4}$$

where $\mu(d)$ is the Möbius function.

Taking into account the results of Ch. XII from ([4]), we conclude that relation (4) is an asymptotic equality for any $0 \leq \gamma \leq \frac{1}{k}$. It is important here that for $\gamma < \frac{1}{k}$ the remainder term, in comparison with the main term, has a power decrease in the order of growth.

It is clear that analogous theorems can also be formulated for sums of the function $\tau_t(Du + l)$ for odd t. Let us note that for no value of $\sigma = 1 - \frac{1}{k}$, $k \geq 3$ (for $k = 2$ see ([2])), is it possible to retain the estimate of Theorem 1 if the exponent $2k$ is increased even by one. In other words, for no $k > 1$ is it possible to replace the bound for D indicated in Theorem 2 by one that is at all better.

The preceding theorems concerned the distribution of the numbers $m_1 \cdots m_k$ in primitive arithmetic progressions $Dn + l$, i.e. progressions with $(l, D) = 1$. The general case $(l, D) \geq 1$ is reduced to the case $(l, D) = 1$ by the following theorem 4.

Theorem 4. Let $(l, D) = d_1 d_2$, $(d_1, d_2) = 1$, and, for $d_1 > 1$, let every prime divisor occurring in d_1 also divide $\frac{D}{(l,D)}$, while for $d_2 > 1$

$$\left(d_2, \frac{D}{(l, D)}\right) = 1; \qquad d_2 = p_1^{\nu_1} \cdots p_\lambda^{\nu_\lambda}; \qquad \nu_i \geq 1; \qquad p_i \text{ are primes.}$$

Then, if $d_1 \geq 1,\ d_2 = 1$, then

$$\sum_{\substack{n \leq x \\ n \equiv l \pmod{C}}} \tau_{2k}(n) = \tau^{2k-1}(d_1) \sum_{\substack{n \leq \frac{x}{d_1} \\ n \equiv \frac{l}{d_1} \pmod{\frac{D}{d_1}}}} \tau_{2k}(n);$$

whereas if $d_1 \geq 1$, but $d_2 \geq 2$, then

$$\sum_{\substack{n \leq x \\ n \equiv l \pmod{D}}} \tau_{2k}(p) = \tau^{2k-1}(d_1) \sum_{s_1=0}^{\nu_1-1} \sum_{j_1=0}^{2k}{}' \cdots \sum_{s_\lambda=0}^{\nu_\lambda-1} \sum_{j_\lambda=0}^{2k}{}' (-1)^{j_1+\cdots+j_\lambda-\lambda} \times$$

$$\times \prod_{i=1}^{\lambda} C_k^{j_i} (s_i+1)^{2k-1} \sum_{\substack{n \leq \frac{x}{d_1 d'} \\ n \equiv l' \pmod{\frac{D}{d_1 d_2}}}} \tau_{2k}(n),$$

where $\tau(d_1)$ is the number of divisors of d_1; the prime on the summation sign indicates that, for the value $s_i = 0$, summation over j_i begins with $j_i = 1$;

$$d' = \prod_{i=1}^{\lambda} p_i^{s_i+j_i}; \qquad l' = \frac{l}{d_1} \prod_{i=1}^{\lambda} p_i^{-s_i-j_i};$$

$p_m^{-s_m-j_m}$ denotes the number inverse to $p_m^{s_m+j_m}$ modulo

$$D\left(d_1 \prod_{i=1}^{m} p_i^{\nu_i}\right)^{-1}; \qquad C_k^j = \frac{k!}{j!(k-j)!}.$$

It is not difficult to see that

$$\left(\frac{l}{d_1}, \frac{D}{d_1}\right) = 1, \quad \text{if } d_2 = 1; \qquad \left(l', \frac{D}{d_1 d_2}\right) = 1, \quad \text{if } d_2 > 1.$$

Thus, for D satisfying inequality (3), the combination of Theorems 3 and 4 gives a solution to the question of the distribution of the numbers $m_1 \dots m_{2k}$ in any progression $Dn + l$ of difference D.

The theorems are derived by means of appropriate generalizations of the method of Yu. V. Linnik's work ([1]). It is based on estimates of L-functions in the critical strip through a special kind of truncated functional equations, differing depending on whether $|t| < 1$ or $|t| \geq 1$. One such equation is given below. In addition, the well-known approximate functional equations of Hardy and Littlewood for the Riemann ζ-function are used throughout, as well as upper estimates for sums of the function $\tau_k(n)$ over intervals of short arithmetic progressions.

In view of the fact that the full proof of the theorems is cumbersome (see ([1])), we shall note here only two of its points, which are also of independent interest.

1. For the values $|t| < 1$, the derivation of Theorem 1 is based on the following lemma. **Lemma.** *For a primitive character χ modulo D, with $s = \sigma + it$, $1/2 \leq \sigma < 1$, the equality holds*

$$L(s,\chi) = \Gamma^{-1}\left(\frac{s+a}{2}\right)\sum_{n=1}^{\infty}\chi(n)n^{-s}\Psi\left(s,a,n\sqrt{\frac{\pi}{D}}\right)$$

$$-\varepsilon(\chi)\left(\frac{D}{\pi}\right)^{1/2-s}\Gamma^{-1}\left(\frac{s+a}{2}\right)\sum_{n=1}^{\infty}\overline{\chi}(n)n^{s-1}\Psi\left(1-s,a,n\sqrt{\frac{\pi}{D}}\right), \qquad (5)$$

where Γ is the gamma-function; $a = \frac{1}{2}(1-\chi(-1))$; $\varepsilon(\chi)$ takes different values only according as $\chi(-1) = 1$ or $\chi(-1) = -1$; $|\varepsilon(\chi)| \leq 1$, and

$$\Psi\left(\omega,a,n\sqrt{\frac{\pi}{D}}\right) = \frac{1}{2\pi i}\int_{2-\sigma-i\infty}^{2-\sigma+i\infty}\Gamma\left(\frac{z+\omega+a}{2}\right)\left(n\sqrt{\frac{\pi}{D}}\right)^{-z}\frac{dz}{z}.$$

The meaning of relation (5) is that, for $|\omega| < c$, the function $\left|\Psi\left(\omega,a,n\sqrt{\pi/D}\right)\right|$ is bounded for all values $n \geq 1$ and rapidly decreases as n grows. Namely, for $n \geq 3\sqrt{D/\pi}$ we have

$$\left|\Psi\left(\omega,a,n\sqrt{\frac{\pi}{D}}\right)\right| = B\exp\left(-c'n\sqrt{\frac{\pi}{D}}\ln n\sqrt{\frac{\pi}{D}}\right),$$

where c, c' are constants.

This lemma is a generalization, from $\sigma = \frac{1}{2}$ to $1/2 \leq \sigma < 1$, of Lemma 7 of Yu. V. Linnik ([1]). Its proof essentially relies on the functional equation of $L(s,\chi)$.

2. For the derivation of Theorem 2, putting

$$M_1(x;D,l) = \sum_{\substack{n\leq x\\ n\equiv l \pmod D}}\tau_{2k}(n); \qquad M_j(x;D,l) = \int_0^x M_{j-1}(y;D,l)\,dy,$$

we obtain

$$M_3(x; D, l) = \frac{1}{2\pi i\varphi(D)} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+2}}{s(s+1)(s+2)} \sum_{\chi \bmod D} \overline{\chi}(l) L^{2k}(s,\chi)\, ds + Bx^2.$$

We shift the contour of integration to the line $\mathcal{R}s = 1 - \frac{1}{k}$. Taking into account the pole of the function $L(s, \chi_0)$ at the point $s = 1$ and applying Theorem 1, we shall have

$$M_3(x; D, l) = \frac{1}{2\pi i\varphi(D)} \oint_C \frac{x^{s+2}}{s(s+1)(s+2)} L^{2k}(s,\chi_0)\, ds = Bx^{3-\frac{k}{k}} \ln^b D,$$

where C is the circle $|s-1| = \frac{1}{k^2 \ln D}$, b is a constant depending only on k. Further, as in (1), one descends from M_3 to M_1.

Here one must act differently in choosing the quantities D, Δ, and Δ'; otherwise for D we shall have a bound close to $x^{1/k}$, but the remainder term will differ from the main term only by a quantity of order $\exp(-\ln^{c''} \ln x)$, where c'' is a constant. Meanwhile, there are problems for the consideration of which it is enough to have uniformity in the distribution of the numbers $m_1 \dots m_{2k}$ in progressions $Dn + l \le x$ of difference $D \le x^{\alpha}$, where α is a constant $< \frac{1}{k}$, but in return a power saving of the remainder term compared with the main term of growth is required. It turns out that results of this kind can be obtained, and this is the special meaning of the generalizations of the theorem of Yu. V. Linnik.

Namely, if for $0 \le \gamma \le \frac{1}{k}$ one chooses δ to be a constant > 1;

$$D \le x^{\gamma} \exp(-\ln^{48} \ln x); \qquad \Delta = x^{\gamma/2-1/2k} \exp(2\ln^{\delta} \ln x); \qquad \Delta' = \sqrt{\Delta},$$

then we obtain Theorem 2.

Received
11 VII 1963

REFERENCES

[1] Yu. V. Linnik, *Matem. sbornik*, 53 (95), No. 1, 3 (1961).
[2] Yu. V. Linnik, *The dispersion method in additive problems*, L., 1961.
[3] Yu. V. Linnik, *Izv. AN SSSR, ser. matem.*, 24, 5, 629 (1960).
[4] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, IL, 1953.