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Abstract

Full Text

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On Entire Analytic Solutions of Infinite-Order Equations with Polynomial Coefficients

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§ 1. The differential equation of infinite order with constant coefficients

$$\sum_{k=0}^{\infty} a_k y^{(k)}(x) = f(x) \tag{1}$$

in the case when the characteristic function $\omega(x) = \sum_{k=0}^{\infty} a_k x^k$ has a nonzero radius of analyticity

$$R = 1/\lim_{k\to\infty} \sqrt[k]{|a_k|},$$

has been studied in considerable detail in the class of exponential solutions ([1,2]). Namely, the following facts are known:

1. Equation (1) is solvable for any exponential function f of type $< R$ in the class M_R of exponential functions of type $< R$; moreover, for any right-hand side f from M_R there is a particular solution of the same order and type as f.
2. The homogeneous equation (with $f \equiv 0$) has in M_R ν linearly independent solutions, where ν is the number of zeros of $\omega(x)$ in the disk $|x| < R$.
3. To each zero λ of multiplicity m of the function $\omega(x)$ there correspond m linearly independent exponential solutions whose type is equal to $|\lambda|$.
4. If λ_0 is the modulus of the zero of $\omega(x)$ nearest to the origin and M_{λ_0} is the class of exponential functions of type $< \lambda_0$, then equation (1) has a unique solution in M_{λ_0} for any f from M_{λ_0}.

For the equation with polynomial coefficients

$$\sum_{k=0}^{\infty} P_k(x) y^{(k)}(x) = f(x) \tag{2}$$

under the condition that $P_k(x)$ are polynomials of fixed degree,

$$P_k(x) = \sum_{s=0}^{p} a_s^k x^s, \qquad \lim_{k\to\infty} \sqrt[k]{|a_s^k|} < \infty, \quad s = 0, 1, \dots, p,$$

lawful results were also obtained ([3,4]) concerning solvability of the equation in the class of exponential functions and the number of arbitrary constants in the solution.

Recently, the theory of equations (2) with polynomial coefficients of increasing degree has begun to be developed. In papers ([5–7]) it is shown that every equation of the form (2), where $P_0(x) \equiv 1$, and $P_k(x)$ is a polynomial of degree $\leq k-1$, $k = 1, 2, \dots$, has its own class of unique solvability, consisting of entire functions of sufficiently small growth. However, the methods of these papers did not allow one to investigate such an equation in broader classes, nor to study the structure of the general solution of the homogeneous equa-

In the present article, one rather general class of equations (2) is singled out, for which it has been possible to obtain results of the same regular character as in the case of the equation with constant coefficients (1).

§ 2. Consider equation (2), where

$$P_0(x) \equiv a_0 \neq 0; \qquad P_k(x) = \sum_{s=0}^{n_k} a_s^k x^s, \quad k = 0, 1, \dots,$$

$$\sup_{k\geqslant 1} \frac{n_k}{k} = \alpha, \quad 0 \leqslant \alpha < 1. \tag{3}$$

In what follows, by $[\rho, \sigma)$ we shall denote the class of entire functions either of order $< \rho$, or of order ρ and type $< \sigma$. Suppose that for at least one $Q > 0$

$$A(Q) = \sum_{k=1}^{\infty} \sum_{s=0}^{n_k} \frac{|a_s^k|\, Q^{k-s}}{([\alpha k] - s)^{\frac{1}{1-\alpha}}} < \infty, \tag{4}$$

where $[x]$ is the symbol for the integer part of x. Let $Q_0 = \sup\{Q : A(Q) < \infty\}$ and

$$E_0 = \left[1-\alpha, \frac{Q_0^{1-\alpha}}{1-\alpha}\right).$$

Introduce the function $\omega(x)$, equal to

$$\sum_{k=0}^{\infty} a_{pk}^{qk} x^{k(q-p)}$$

if α is rational, $\alpha = p/q,\ p < q,\ (p,q) = 1$, and equal to a_0 for irrational α. Let γ be the modulus of the zero of $\omega(x)$ nearest to the origin, and let $\gamma_1 = \min\{\gamma, Q_0\}$. Finally, denote by E_1 the class

$$\left[1-\alpha, \frac{(\gamma_1)^{1-\alpha}}{1-\alpha}\right).$$

For equation (2), under conditions (3) and (4), the following results are valid.

Theorem 1. For any right-hand side f from E_0 there exists a particular solution of equation (2) of the same order and type as f.

Theorem 2. The homogeneous equation (with $f \equiv 0$) has in E_0 ν linearly independent solutions, where ν is the number of zeros of $\omega(x)$ in the disk $|x| < Q_0$; moreover, to each group of m zeros (counting multiplicities) lying on the circle $|x| = R < Q_0$, there correspond m linearly independent solutions of the homogeneous equation of growth of exact order $1-\alpha$ and type $R^{1-\alpha}/(1-\alpha)$.

Theorem 3. Equation (2), for any function f from E_1, has a unique solution in E_1.

For $\alpha = 0$, Theorems 1-3 imply the known results for the equation with constant coefficients stated in § 1. Let us also note that it has not been possible to apply to equation (2), under conditions (3) and (4), the methods of works ([1–8]) (in work ([7]), instead of condition (4), a much stronger assumption is made that the function $\sum_{n=1}^{\infty} n! P_n(x) t^{-n}$ is analytic in some bicylinder $|x| \leqslant R,\ |t| \geqslant R_1$; moreover, only a theorem on unique solvability in a rather narrow subclass of functions from E_1 is established there).

We indicate the idea of the proofs of Theorems 1-3. Equation (2) is replaced by the equivalent (under conditions (3) and (4)) infinite system of linear algebraic equations for the Taylor coefficients of the solution. After a number of substitutions, the latter system is transformed so that one can apply to it the theory developed in the works of M. G. Krein and his students ([9–10]). The system of equations is studied first in the Banach spaces l_r, $r \geqslant 1$ (here the results of works ([9], [10]) are applied), and then in the inductive limits of such spaces—in analytic spaces of sequences of Taylor coefficients of functions from E_0 and E_1.

A convergent process has also been found for the approximate determination of all nontrivial solutions of the homogeneous equation (2) (with $f \equiv 0$) from E_0, as well as

of a particular solution of the nonhomogeneous equation for any right-hand side f from E_0. An approximate solution from E_1 is found especially simply if $f \in E_1$. In this case, if f is an entire function of order ρ and type σ, and $y_n(x)$ is the sequence of polynomials $y_n(x)$ obtained by the "truncation" method described in (5), then $y_n(x) \to y(x)$ uniformly in the whole plane with weight $\psi_\varepsilon(|x|)$, where $\psi_\varepsilon(r) = \exp[-(\sigma+\varepsilon)r^\rho]$, if $\sigma < \infty$, and $\psi_\varepsilon(r) = \exp(-\varepsilon r^{\rho+\varepsilon})$, if $\sigma = \infty$, while $\varepsilon > 0$ is a fixed number, which may be taken arbitrarily small.

§ 3. As Valiron showed ([11,12]), every entire transcendental solution of an equation of finite order of the form

$$\sum_{k=0}^{n} P_k(x) y^{(k)}(x) = Q_0(x),$$

where Q_0 and P_k are polynomials, has rational order and belongs to an exponential star type ([12]); in other words, for some rational b there exists

$$\lim_{r\to\infty} r^{-b} \ln M(r, y),$$

where

$$M(r, y) = \max_{|x|=r} |y(x)|,$$

and, moreover, the set of arguments of those points x_r on the circle $|x| = r$ at which $|y(x_r)| = M(r, y)$ has a finite number of limit points when r runs through all exceptional ([12]) values.

Suppose that the coefficients $P_k(x)$ of equation (2) satisfy condition (3) and the following condition (more stringent than (4)):

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|c_k|} = \frac{1}{c} < \infty; \qquad c_k = \sup_{s\le n_k} |a_s^k|, \quad k = 1, 2, ... \tag{5}$$

This condition is equivalent (under the assumption that $n_k \le \alpha k,\ k = 1, 2, ...$) to the requirement of analyticity of the function

$$\sum_{n=1}^{\infty} P_n(x) t^{-n}$$

in some bicylinder $|x| \le R,\ |t| \ge R_1$.

Theorem 4. *Let in the equation*

$$\sum_{k=0}^{\infty} P_k(x) y^{(k)}(x) = Q_0(x) \tag{6}$$

Q_0 and P_k be polynomials, $P_k(x)$ satisfy conditions (3) and (5), and α be rational. Then every transcendental entire solution from the class

$$E_2\left[1-\alpha, \frac{c}{e(1-\alpha)}\right)$$

is an exponential star function; for each such solution

$$\lim_{r\to\infty} \ln M(r, y) = \frac{(\gamma_0)^{1-\alpha}}{1-\alpha},$$

where γ_0 is one of the zeros of $\omega(x)$.

Remark. Theorem 4 is substantive if $E_1 \subset E_2$; it is not difficult to indicate conditions under which such a strict inclusion takes place.

Valiron' s results for an equation of finite order were obtained by means of the exact estimates he found, relating $M(r, y^{(k)})$ and $M(r, y)$. However, these estimates are valid only for a given fixed k. The proof of Theorem 4 is rather complicated and is based on combining Valiron' s exact estimates with the coarser estimates, obtained by the author, of the quantities $M(r, y^{(k)})$ (in terms of $M(r, y)$), which are already valid for all $r \geq 1$ and $k \geq 1$. Apparently, Theorem 4 is valid (under conditions (3) and (5)) for the class

$$\left[1-\alpha, \frac{c}{1-\alpha}\right).$$

§ 4. The results obtained above find application in the study of certain problems of the analytic theory of partial differential equations, and also in the solution of functional equations. Thus, with their aid the results of works ([13,14]) are significantly strengthened. Let us indicate some further applications of Theorems 1–4.

1. Consider the equation

$$y\left(x+a\sqrt{x}\right)+y\left(x-a\sqrt{x}\right)=f(x), \tag{7}$$

where a is a complex number and $f(x)$ is an entire function. Let B be the class $[1/2, \infty)$, and B_1 the class $[1/2, \pi/|a|)$. From Theorems 1-4 there follow the following results for entire solutions of equation (7).

For any right-hand side f from B, equation (7) has a particular solution of the same order and type as f. If $f \in B_1$, then the equation has a unique solution in B_1.

The homogeneous equation $y(x+a\sqrt{x})+y(x-a\sqrt{x})=0$ has in B a countable number of linearly independent solutions; all of them belong to exponential star type, and there exists

$$\lim_{r\to\infty} r^{-1/2}\ln M(r, y)=\frac{\pi}{|a|}(2l+1),$$

where l is equal to one of the natural numbers.

2. Let us also consider the following problem: to find an entire harmonic function $u(x, y)$ from its values on the parabola $y = \pm a\sqrt{x}$, $a > 0$:

$$u(x, a\sqrt{x})=\varphi_1(x), \qquad u(x, -a\sqrt{x})=\varphi_2(x). \tag{8}$$

It is easy to show that, if the problem is solvable, then the functions φ_1 and φ_2 admit the representation

$$\varphi_1(x) = v_1(x) + \sqrt{x}\, v_2(x), \qquad \varphi_2(x) = v_1(x) - \sqrt{x}\, v_2(x), \tag{9}$$

in which v_1, v_2 are entire functions with real Taylor coefficients. From Theorems 1-3 it follows that if φ_1 and φ_2 admit the representation (9) and $v_1, v_2 \in B$, then the posed problem is always solvable. Denote by B_2 the class $[1/2, 2\pi/|a|)$. Then the following result is valid: if $v_1 \in B_1$, $v_2 \in B_2$, then the problem has a unique solution in the class of entire harmonic functions such that $u(x,0) \in B_1$, $u'_y(x,0) \in B_2$. As a consequence we obtain an assertion of the type of theorems of Phragmén–Lindelöf:

Theorem 5. If an entire harmonic function $v(x,y)$ is such that: a) $v(x, a\sqrt{x}) = v(x, -a\sqrt{x}) = 0$; b) $v(x,0) \in [1/2, \pi/|a|)$; c) $v'_y(x,0) \in [1/2, 2\pi/|a|)$, then $v(x,y) \equiv 0$.

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Note: Figure translations are in progress. See original paper for figures.