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Abstract

Full Text

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ON THE HOMOTOPIC TOPOLOGY OF BANACH SPACES

(Presented by Academician P. S. Aleksandrov on 17 VII 1963)

In the first part of the present note a homotopic classification of certain types of mappings of Banach spaces is indicated. The second part of the note is devoted to the study of homotopic properties of certain spaces of linear operators; the results formulated in it will be useful in the homotopic classification of certain classes of differentiable mappings of Banach spaces.

1. Let E and F be Banach spaces. Fix a continuous Φ -operator A of index k (¹), acting from E to F . Let K be a ball in E , and L the boundary of the ball K . A continuous mapping G of the ball K into the space F will be called **admissible** if the set $G(L)$ does not contain the point 0, and the set of points of the form $Gx - Ax$, where $x \in K$, is compact (i.e., the operator $G - A$ is completely continuous). Two admissible mappings G_0 and G_1 are called **homotopic** if there exists a family G_t ($0 \leq t \leq 1$) of admissible mappings joining them such that $G_t(x)$ is a continuous function of two variables $x \in K$, $t \in [0; 1]$, and the set of points of the form $G_t x - Ax$, where $x \in K$, $0 \leq t \leq 1$, is compact. The set of homotopy classes of admissible mappings of the ball K into F will be denoted by $S(A)$.

Theorem 1. *A one-to-one correspondence can be established between the set of homotopy classes $S(A)$ and the k -th stable homotopy group of the sphere $G_k = \pi_{n+k}(S^n)$, where $n > k$.*

If the operator A fixed by us is taken to be the identity, then Theorem 1 reduces to the theorem, obtained by Leray–Schauder–Rothe (²), on the homotopic classification of completely continuous vector fields.

Let now K be a metric space, L its closed subset, and A a proper mapping of the space K onto a closed subset of the Banach space F (i.e., a continuous mapping under which the preimage of every compact set is compact). Admissible mappings and the homotopy of admissible mappings are defined in the same way as above. Consider the problem of the homotopic classification of admissible mappings in this more general case.

Suppose that the Banach space F has a basis f_1, \dots, f_n, \dots , and denote by F_n the subspace consisting of vectors of the form $\lambda_1 f_1 + \dots + \lambda_n f_n$, and by F_n^+ (F_n^-) the half-space of the space F_n composed of vectors for which $\lambda_n \geq 0$ ($\lambda_n \leq 0$).

Consider the sets

$$V_n = A^{-1}(F_n) \cap L, \quad V_n^+ = A^{-1}(F_n^+) \cap L, \quad V_n^- = A^{-1}(F_n^-) \cap L$$

and denote by Π_n the set of homotopy classes of mappings of the space V_n into the $(n-1)$ -dimensional sphere S^{n-1} . Define the mapping $i_n : \Pi_n \rightarrow \Pi_{n+1}$ by putting $i_n(\alpha) = \beta$, if there are mappings $f \in \alpha, g \in \beta$ such that $f = g$ on $V_n, g(V_{n+1}^+) \subset S_+^n, g(V_{n+1}^-) \subset S_-^n$ (the sphere S^{n-1} is regarded as the equator of the sphere S^n ; S_+^n and S_-^n are the upper and lower hemispheres of the sphere S^n). It is not difficult to check that this definition is correct.

Theorem 2. *A one-to-one correspondence can be established between the set $S(A)$ of homotopy classes of admissible mappings and the limit of the direct spectrum of the sets $\{\Pi_n, i_n\}$.*

We shall call a mapping G of the ball K of a Banach space E into a Banach space F a **k -mapping** if it can be represented in the form $G = A + R$, where A is a continuous Φ -operator of index k , and R is a completely continuous operator; a k -mapping will be called **admissible** if the set $G(L)$, where L is the boundary of the ball K , does not contain 0. Two admissible k -mappings G_0 and G_1 are regarded as homotopic if there exist families of operators A_t and R_t ($0 \leq t \leq 1$) such that $G_0 = A_0 + R_0, G_1 = A_1 + R_1, G_t = A_t + R_t$ are admissible mappings ($0 \leq t \leq 1$); A_t is a family of continuous Φ -operators continuously depending on t (in the sense of the usual norm in the space of linear operators); $R_t(x)$ is a continuous function of two variables $x \in K, t \in [0, 1]$; the set of points of the form $R_t(x)$, where $x \in K, 0 \leq t \leq 1$, is compact. We shall denote the set of homotopy classes of admissible k -mappings by Σ_k .

Theorem 3. *If F is a Hilbert space, then the natural mapping of the set $S(A)$ (A is a Φ -operator from E to F of index k) into the set Σ_k is a mapping "onto"; two elements x, y of the set $S(A)$ pass into one and the same element of the set Σ_k if and only if $x = -y$.*

(By virtue of the one-to-one correspondence established by Theorem 1, the set $S(A)$ is endowed with a group structure.)

Every 0-mapping can be represented in the form $G = A + R$, where A is an invertible operator; two admissible 0-mappings $G_0 = A_0 + R_0, G_1 = A_1 + R_1$, represented in this way, are homotopic if and only if the degrees of the mappings

$$A_0^{-1}G_0 = I + A_0^{-1}R_0$$

and

$$A_1^{-1}G_1 = I + A_1^{-1}R_1$$

are equal in absolute value.

2. Let E be a Banach space over the field of real numbers; $L(E)$ the Banach space of linear bounded operators acting in E , with the usual norm. By $GL(E)$ we shall denote the subset of $L(E)$ consisting of invertible operators; by $GL_c(E)$

the subset of $GL(E)$ consisting of operators of the form $I + R$, where I is the identity and R is a completely continuous operator; by $R_k(E)$ we shall denote the subset of $L(E)$ consisting of Φ -operators of index k .

The spaces $GL(E)$ and $GL_c(E)$ are topological groups with respect to multiplication of operators; if E is finite-dimensional, then both of these groups coincide with the full linear group, whose homotopy properties (coinciding with the homotopy properties of the orthogonal group $O(n)$) are well studied.

By O , as usual ⁽³⁾, we shall denote the union of the orthogonal groups $O(n)$ ($n = 1, 2, \dots$), naturally embedded one into another.

We shall say that a Banach space E **satisfies condition** (*) if there exists such a directed set of finite-dimensional linear operators $\{P_\lambda\}$ that

$$\sup_{\lambda} \|P_\lambda\| < \infty$$

and for every $x \in E$ we have

$$\lim_{\lambda} P_\lambda x = x.$$

(This condition is fulfilled if the space E has a basis; no spaces are known for which it would fail.)

Theorem 4. *If an infinite-dimensional space E satisfies condition (*), then the spaces $GL_c(E)$ and O are weakly homotopy equivalent (i.e., there exists a continuous mapping of O into $GL_c(E)$ inducing an isomorphism of homology groups and homotopy groups).*

Recall that the homology groups and homotopy groups of the space O are completely known ⁽³⁾.

If the space E is represented as a direct sum of subspaces H_1 and H_2 , then we denote by $GL(E, H_1, H_2)$ the subgroup of the group $GL(E)$ consisting of transformations that are the identity on H_2 and map H_1 onto itself; this subgroup is obviously isomorphic to the group $GL(H_1)$.

We shall say that a space E **satisfies condition** (**) if it contains an infinite-dimensional subspace E' with sym-

metric* basis that has a complement in E . Let us note that this condition is satisfied by the spaces l^p and \mathcal{L}^p ($1 \leq p < \infty$), c_0 , C , and the Orlicz spaces E_M .

Theorem 5. *If the space E satisfies condition (**), and the space H_1 is finite-dimensional, then the subgroup $GL(E, H_1, H_2)$ is contractible to a point in the space $GL(E)$. If the space E is Hilbert, then the subgroup $GL(E, H_1, H_2)$ is contractible in $GL(E)$ to a point also in the case when both spaces H_1 and H_2 are infinite-dimensional.*

Let us note that if the space H_2 is finite-dimensional and H_1 is infinite-dimensional, then the embedding of $GL(E, H_1, H_2)$ into $GL(E)$ is a homotopy equivalence.

If the space E satisfies both conditions (*) and (**), then the first assertion of Theorem 2 can be slightly strengthened. Namely, then:

Theorem 5'. Every compact subset of the space $GL_c(E)$ is contractible to a point in the space $GL(E)$ (and, consequently, the embedding of $GL_c(E)$ into $GL(E)$ induces the zero homomorphism of homology and homotopy groups).

The study of the homotopy properties of the space $GL(E)$ encounters serious difficulties; as far as the author knows, for an infinite-dimensional space E not isomorphic to Hilbert space it is not even clear whether the space $GL(E)$ is connected (if E is a Hilbert space, then $GL(E)$ is connected). Theorem 5' permits one to formulate the hypothesis that, as a rule, for an infinite-dimensional space E the space $GL(E)$ is aspherical in all dimensions.

Theorem 6. The space $R_0(E)$ is weakly homotopy equivalent to the quotient group $GL(E)/GL_c(E)$; if the space E satisfies conditions (*) and (**), then the embedding $i : GL(E) \rightarrow R_0(E)$ induces a monomorphism of homotopy groups, and the quotient group of the group $\pi_k(R_0(E))$ by the subgroup $i_*\pi_k GL(E)$ is isomorphic to the group $\pi_{k-1}(O)$.

Theorem 7. If the space $R_k(E)$ is nonempty, then it is homotopy equivalent to the space $R_0(E)$.

Let us note that in order for the space $R_k(E)$ to be nonempty, it is necessary and sufficient that the Banach space E be isomorphic to the direct sum $E + N$, where N is a space of dimension $|k|$; Banach conjectured that this condition is fulfilled for all Banach spaces ⁽⁴⁾.

Remark 1. The transfer of the theorems formulated above to the case of a Banach space over the field of complex numbers presents no difficulties; the role of the group O is then played by the group U .

Remark 2. If E is a Hilbert space, then the space $GL(E)$ is homotopy equivalent to the space $I(E)$ of isometrically invertible operators acting in E , and the space $GL_c(E)$ is homotopy equivalent to the space $I_c(E) = GL_c(E) \cap I(E)$.

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* We shall call a basis **symmetric** if every permutation of the elements of the basis can be extended to an isometry of the entire space.

Note: Figure translations are in progress. See original paper for figures.

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