MATHEMATICS

P. S. BONDARENKO

1964

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Abstract

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P. S. BONDARENKO

COMPUTATIONAL ALGORITHMS FOR THE APPROXIMATE SOLUTION OF OPERATOR EQUATIONS

(Presented by Academician A. A. Dorodnitsyn, 29 X 1962)

Let abstract spaces M_1, M_2 and an operator A, which maps elements of M_1 into elements of M_2, be given. It is required, for a given element $y \in M_2$, to find an element $x \in M_1$ such that the equality

$$Ax = y. \tag{1}$$

is satisfied.

Suppose that the spaces M_1, M_2 and the operator A are replaced by some other spaces $\overline{M}_1, \overline{M}_2$ and by an operator $\overline{A}$, which maps elements of $\overline{M}_1$ into elements of $\overline{M}_2$, and suppose that instead of the original problem the following problem is solved: for an element $\overline{y} \in \overline{M}_2$, it is required to find an element $\overline{x} \in \overline{M}_1$ such that the equality

$$\overline{A}\overline{x} = \overline{y}. \tag{2}$$

is satisfied.

Denote by $N(\overline{A})$ the aggregate of computational operations that make it possible, from the known element $\overline{y} \in \overline{M}_2$, to construct such an element $\overline{x} \in \overline{M}_1$ that equality (2) is satisfied. If the existence of an operator $\overline{A}^{-1}$, inverse with respect to $\overline{A}$, is assumed, the symbol $N(\overline{A})$ denotes the method of its actual construction:

$$N(\overline{A})\overline{y} = \overline{x}. \tag{3}$$

Let, further, in the course of carrying out the method $N(\overline{A})$, by means of computing devices computations be performed with $s + r$ $(r \geq 0)$ digits after the decimal point, and let the result of computing the element $\overline{x}$ be rounded to the s-th digit after the decimal point. We shall denote the element $\overline{x}$ thus computed by $\overline{x}^*$. The aggregate of computational operations performed in the course of such an implementation of the method $N(\overline{A})$ will be denoted by $N^*(\overline{A})$.

The process consisting of methods of approximating the spaces M_1, M_2 by the spaces $\overline{M}_1, \overline{M}_2$, the operator A by the operator $\overline{A}$, of replacing equation (1) by equation (2), together with the method $N(\overline{A})$ of solving the latter equation, will be called a **computational algorithm for solving equation** (1) and denoted by $A_N(\overline{x})$. A computational algorithm $A_N(\overline{x})$ in which the method $N(\overline{A})$ is replaced by the method $N^*(\overline{A})$ will be called a **real computational algorithm for solving equation** (1) and denoted by $A_{N^*}(\overline{x}^*)$.

In the present note the basic properties of computational and real computational algorithms for the solution of one sufficiently broad class of linear operator equations are considered.

1. Let M_1, M_2 be linear spaces with norms $\| \ \|_{M_1}, \| \ \|_{M_2}$. Let, further, A and $\overline{A}_n$ $(n = 1, 2, ...)$ be linear operators mapping elements of M_1 into elements of M_2, and let equation (1)

$$Ax = y \tag{4}$$

be replaced by the equation

$$\overline{A}_n x_n = y, \qquad x_n \in \overline{X}_n \subset M_1. \tag{5}$$

Let the approximation be characterized by the relation

$$\lim_{n\to\infty} \|Ax - \bar{A}_{nx}\|_{M_2} = 0 \tag{6}$$

or by the relation

$$\|Ax - \bar{A}_{nx}\|_{M_2} \leq Mn^{-k}, \tag{7}$$

where $M > 0$ is a constant independent of n, valid for any x from M_1.

Suppose that in the space M_1 there is singled out an everywhere dense subspace $\bar{X}_n$, appearing in (5), so that the solution $\bar{x}_n$ of equation (4) corresponding to a given value of the parameter n belongs to $\bar{X}_n$: $x_n \in \bar{X}_n$. Then the convergence of the solution $\bar{x}_n$ of equation (5) to the solution x_n of equation (4) is determined by the relation

$$\lim_{n\to\infty} \|x_n - \bar{x}_n\|_{\bar{X}_n} = 0. \tag{8}$$

Definition 1. Suppose that equation (4) has a unique solution $x \in M_1$ for a given $y \in M_2$. If the operators $\bar{A}_n$ are constructed so that equation (5) has a solution $\bar{x}_n$ for every n, $1 \leq n < \infty$, and arbitrary $y \in M_2$, depending continuously on y, with this continuous dependence uniform in n, and if this solution, as $n \to \infty$, converges to the solution x_n of equation (4), then the computational algorithm $A_N(\bar{x}_n)$ is called **regularly convergent**. If the operators $\bar{A}_n$ are constructed so that equation (5) has a unique solution $\bar{x}_n$ for all n and $y \in M_2$, depending continuously on y, but this continuous dependence is not uniform in n, then, provided $\bar{x}_n$ converges to x_n, the computational algorithm $A_N(\bar{x}_n)$ is called **convergent**. An element $\bar{x}_n$ found by a regularly convergent (convergent) computational algorithm is called a **regular approximate** (**approximate**) solution of equation (4). An element $\bar{x}_n^*$ found by the real computational algorithm $A_{N^*}(\bar{x}_n^*)$ is called a **real approximate solution** of equation (4).

Definition 2. The real computational algorithm $A_{N^*}(\bar{x}_n^*)$ for solving equation (4) is called **stable** if there exists an integer $r_n^*(s) \geq 0$ such that the inequality

$$a_n(s,r) = \|\bar{x}_n - \bar{x}_n^*\| \leq \frac{1}{2}\,\omega\beta^{-s}, \tag{9}$$

where $\beta > 0$ is the base of the number system, $\frac{1}{2} \leq \omega \leq 1$, is satisfied for any independently fixed n and s, $0 \leq s < \infty$, $1 \leq n < \infty$, and for all r, $r_n^*(s) \leq r < \infty$, in the process of computation by the algorithm $A_{N^*}(\bar{x}_n^*)$ with $r+s$ digits after the decimal point. The integer $r_n^*(s)$ is called the **characteristic of the stability region of the algorithm** $A_{N^*}(\bar{x}_n^*)$.

Remark. For the computational algorithm $A_N(\bar{x}_n)$, for any fixed n and s, the characteristic of its stability region is $r_n^*(s) = \infty$ by definition.

Let x_n be the solution of equation (4), and let $\bar{x}_n^*$ be a real approximate solution of this equation. We call the difference $x_n - \bar{x}_n^*$ the **true error of the real approximate solution** $\bar{x}_n^*$. If the computational algorithm $A_N(\bar{x}_n)$ is convergent (regularly convergent) and the number s is given: $s = s^*$, then there exists a natural number n_0 such that for all $n \geq n_0$ the inequality

$$\|x_n - \bar{x}_n^*\| \leq \frac{1}{2}\,\omega\beta^{-s^*}. \tag{10}$$

If the real computational algorithm $A_{N^*}(\bar{x}_n^*)$ is stable, then there exists an integer $r_{n_0}^*(s^*)$ such that the inequality

$$\left\|\bar{x}_{n_0} - \bar{x}_{n_0}^*\right\| \leqslant \frac{1}{2}\omega\beta^{-s^*}. \tag{11}$$

holds.

From (10) and (11), obviously, the inequality follows

$$\left\|x_{n_0} - \bar{x}_{n_0}^*\right\| \leqslant \omega\beta^{-s^*}. \tag{12}$$

Definition 3. Let the means of computation by the algorithm $A_{N^*}(\bar{x}_n^*)$ be known. If, for the given $s = s^*$, computations with $s^* + r_{n_0}^*(s^*)$ digits after the decimal point are feasible with the aid of these means, then we call the computational algorithm $A_{N^*}(\bar{x}_n^*)$ **really stable**.

2. The following theorems contain criteria for regularly convergent, convergent computational algorithms, as well as criteria for stable real computational algorithms.

Theorem 1. Suppose:

1) equation (4) has a unique solution from M_1 for the given y from M_2;
2) equation (5) approximates equation (4) on elements x from M_1;
3) the operators $\bar{A}_n$ $(n = 1, 2, ...)$ have inverses, bounded in the aggregate:

$$\left\|\bar{A}_n^{-1} y\right\|_{\bar{X}} \leqslant K \|y\|_{M_2};$$

4) the operator $\bar{A}_n$ is such that for any $\bar{x}_n$ from its domain the inequality is fulfilled

$$\left\|\bar{A}_n \bar{x}_n\right\|_{\bar{X}_n} \leqslant C n^m \left\|\bar{x}_n\right\|_{\bar{X}_n},$$

where $C > 0$ is a constant independent of n and $\bar{x}_n$, and $m \geqslant 0$ is an integer;

5) the method $N^*(\bar{A}_n)$ is characterized by the relation

$$\alpha_n(s, r) = \left\|N(\bar{A}_n) y - N^*(\bar{A}_n) y\right\|_{\bar{X}_n} = \|\bar{x}_n - \bar{x}_n^*\|_{\bar{X}_n} \to 0$$

as $s + r \to \infty$ and for any independently fixed s and n.

Then the computational algorithm $A_N(\bar{x}_n)$ is regularly convergent, and the real computational algorithm $A_{N^*}(\bar{x}_n^*)$ is stable.

Theorem 2. Suppose:

1) equation (5) approximates equation (4), and the order of this approximation is $k > 0$;
2) the operators $\bar{A}_n$ $(n = 1, 2, ...)$ have inverses satisfying the condition

$$\left\|\bar{A}_n^{-1} y\right\|_{\bar{X}_n} \leqslant P n^l \|y\|_{M_2},$$

and the number $l > 0$ satisfies the condition $k - l > 0$;

3) conditions 1), 4) (with $m = 0$) and 5) of Theorem 1 are fulfilled.

Then the computational algorithm $A_N(\bar{x}_n)$ is convergent, and the real computational algorithm $A_{N^*}(\bar{x}_n^*)$ is stable.

If the computational means are known, then, on the basis of Theorems 1 and 2, one can single out from the real computational algorithms $A_{N^*}(\bar{x}_n^*)$ the really stable computational algorithms for solving the considered class of operator equations.

Kyiv State University
named after T. G. Shevchenko

Received
25 X 1962

Note: Figure translations are in progress. See original paper for figures.