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MATHEMATICS

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ON SOME CLASSES OF TOPOLOGICAL GROUPS

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Topological groups with various conditions of commutativity and finiteness type have been studied in the works of V. M. Glushkov ([1–4]) and V. S. Charin ([5–7]). In ([3]) the structure was established of topological locally bicompact groups satisfying the minimality condition for closed subgroups, and in ([4]) it was proved that, for locally nilpotent groups, the minimality conditions for closed abelian subgroups and for all subgroups are equivalent. Subsequently V. S. Charin, in papers ([5], [6]), transferred the latter result to locally soluble topological groups, thereby generalizing the known result of S. N. Chernikov for the discrete case ([8]).

In the present paper the following classes of groups are considered: 1) topological groups satisfying the minimality condition for closed abelian subgroups; 2) pure topological groups; 3) complete topological groups. The indicated classes of groups are studied under general assumptions, without imposing conditions of the type of generalized solubility or nilpotency, and their structure is clarified, in a certain sense, completely.

The above-mentioned results of V. M. Glushkov and V. S. Charin, which pertain mainly to groups of the first class, follow from the results of this paper as special cases. In conclusion, some classes of generalized soluble and generalized nilpotent groups are studied. In particular, it is shown that V. M. Glushkov' s results ([1]) on the structure of locally bicompact locally nilpotent (in the abstract sense) groups carry over to the more general case of locally projectively nilpotent groups, i.e. to the case of groups locally nilpotent in the topological sense.

The proofs are based on methods applied by the author in ([9], [10]); in particular, an important role is played by the general Cartan–Mal' cev–Iwasawa theorem from ([9]), and they rely on results of A. I. Mal' cev ([11]), K. Iwasawa ([12]), H. Yamabe ([13], [14]) (see also ([15])), and S. N. Chernikov ([8], [18]). In addition, the study of groups of the first class is based on the study of automorphism groups of compact Lie groups (algebras). In the proofs the theory of projective ([16]) and transfinite ([17]) limits of topological groups is used effectively.

All groups considered below are locally bicompact. The group-theoretic and topological terminology is standard ([16], [17], [20]).

Theorem 1. *A connected topological group satisfying the minimality condition for closed abelian subgroups is a connected compact Lie group.*

It is easy to verify that a connected compact Lie group satisfies the minimality condition for all closed subgroups, i.e. for connected groups the minimality conditions for all subgroups and for abelian subgroups are equivalent.

Let now G be a topological group whose connected component G_0 of the identity is a compact Lie group, and suppose that the factor group G/G_0 is periodic in the discrete sense (a group is called periodic in the discrete sense if all its elements have finite order). Denote by $Z_G(G_0)$ the centralizer of G_0 in G. Then the following is true:

Lemma 1. The group $H = Z_G(G_0)$ has finite index in the group G.

With the aid of Theorem 1, Lemma 1, and the results indicated above, the main

Theorem 2. A topological group G satisfies the minimality condition for closed abelian subgroups if and only if its connected component G_0 is a compact Lie group, and $G^* = G/G_0$ is a group periodic in the discrete sense that satisfies the minimality condition for arbitrary abelian (not necessarily closed) subgroups.

Thus, a topological group G satisfies the minimality condition for closed abelian subgroups if and only if it is an extension of a connected compact Lie group by an abstract group with the minimality condition for abelian subgroups. Although the factor group $G^* = G/G_0$ may be regarded as an abstract group, since closedness of abelian subgroups is not required in our case, there remains a certain inconvenience connected with the fact that the group G^* need not be countable in the natural topology induced by the group G. Under some rather general assumptions this inconvenience can be removed.

Namely, let B^* be an open bicompact subgroup of the group G^*. From entirely general considerations it follows that all abelian subgroups of B^* are finite. Hence B^* is a bicompact periodic group all of whose abelian subgroups are finite. Our assumption is that B^* is a locally finite group; then from a result of M. I. Kargapolov ([19]) it follows that B^* is a finite group, and consequently $G^* = G/G_0$ is a discrete group.

We note that, when conditions of the type of generalized solvability and generalized nilpotency are imposed on the group G, the group B^* will always be locally finite.

We give the following corollaries, which follow from Theorem 2 and the corresponding results of S. N. Chernikov ([8]).

Corollary 1 ([3]). A topological group Γ satisfies the minimality condition for closed subgroups if and only if Γ_0 is a compact Lie group and $\Gamma^* = \Gamma/\Gamma_0$ is a discrete group with the minimality condition for subgroups.

Corollary 2 ([5]). A locally solvable topological group Γ satisfying the minimality condition for closed abelian subgroups satisfies the minimality condition for all closed subgroups. In this case the group Γ possesses a closed abelian subgroup H of finite index, decomposable into the direct product of a finite number of one-dimensional tori and discrete groups of type p^∞.

From the results of B. I. Plotkin ([11]) and Theorem 2 there follows

Theorem 3. A topological nilgroup Φ satisfies the minimality condition for closed abelian subgroups if and only if Φ_0 is a central commutative Lie group (an n-dimensional torus), and the factor group Φ/Φ_0 is a discrete nilgroup with the minimality condition for abelian subgroups.

Theorem 3 and V. G. Vil' jačer' s result ([22]) on local nilpotency of a nilgroup with the minimality condition for subgroups make it possible to formulate the following.

Corollary. A topological nilgroup satisfying the minimality condition for closed subgroups is locally nilpotent.

The structure of locally nilpotent groups with the minimality condition for closed subgroups is well known ([4]), and also follows from Corollary 2.

We turn to the consideration of pure topological groups. An element g of a topological group G is called **bicompact** if the closure of the cyclic subgroup g is bicompact in G. A topological group G is called **pure** (see ([2])), or **without torsion**, if it

does not contain bicompact elements. It is clear that in the discrete case purity is equivalent to the absence in the group G of elements of finite order.

The following theorem completely clarifies the structure of pure topological groups.

Theorem 4. *A topological group G is pure if and only if G_0 is a pure Lie group and G/G_0 is a discrete torsion-free group. A pure Lie group $G_0 = S \cdot R$, $S \cap R = (e)$, where S is a pure semisimple Lie group and R is a simply connected radical. Pure semisimple Lie groups are completely classified; namely, there is only a finite number of pure simple Lie groups up to isomorphism, and all of them are realized in explicit form.*

Thus every pure topological group is a Lie group, and indeed of a very special kind.

Corollary 1 ([2]). *The connected component Γ_0 of a pure locally nilpotent topological monological group Γ is a simply connected nilpotent Lie group, and Γ/Γ_0 is a discrete locally nilpotent torsion-free group.*

From the results of V. S. Charin ([23]) and Theorem 4 there follows

Corollary 2. *A pure topological locally solvable group of finite rank (in the topological sense) is solvable.*

Relying on Theorem 4 and the results of B. I. Plotkin ([21]), one can obtain the following theorem:

Theorem 5. *A pure topological nilgroup of finite rank (in the topological sense) is nilpotent.*

Thus the above-mentioned results of B. I. Plotkin and V. S. Charin are generalized to the case of topological groups.

A group R is called **complete in the sense of S. N. Chernikov** ([18]) if, for every integer n, it is generated by the n-th powers of its elements.

It is quite obvious that every group complete in the sense of extraction of roots is also complete in the sense of S. N. Chernikov, whereas the converse assertion is true for nilpotent groups and is no longer true for arbitrary solvable groups. For topological groups the question of their completeness is very natural, since the notion of completeness introduces into a group, in a certain sense, a certain notion of closeness.

It is not difficult to verify that every connected compact Lie group is a complete group with unrestricted extraction of roots. From the approximation of every compact group by Lie groups there follows

Theorem 6. *A compact group Γ is complete in the sense of extraction of roots if and only if it is connected.*

From the general Cartan–Mal'cev–Iwasawa theorem ([9]) and Theorem 4 it follows that

Theorem 7. *A topological group G is complete in the sense of S. N. Chernikov if and only if the factor group G/G_0 is complete in the sense of S. N. Chernikov.*

Thus, for locally bicompact connected groups, completeness is obtained only in the sense of S. N. Chernikov; moreover, already simple examples show that there exist connected groups which are not complete in the sense of extraction of roots. The simplest example of this kind is provided by the following two-step solvable connected Lie group.

Let T be the group of complex matrices of the form $\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\right\}$, where $\alpha\gamma = 1$. It is obvious that T is a connected solvable Lie group; however, it is easy to check that it is not complete in the sense of extraction of roots. The group T also provides an example of a solvable Lie group which is not covered by its one-parameter subgroups. At the same time, for nilpotent groups one can note the following

Corollary. *A connected topological nilpotent group is complete in the sense of extraction of roots.*

A topological group G is called **projectively solvable** (**nilpotent**) if in every neighborhood U of its identity there is a normal divisor H_U such that the factor group G/H_U is solvable (respectively nilpotent). If every finitely generated subgroup of a topological group Γ is projectively nilpotent (solvable), then Γ is called **locally projectively nilpotent** (**solvable**).

From the general Cartan–Mal' cev–Iwasawa theorem ([9]) and the corresponding results for linear groups there follows

Theorem 8. *If a connected topological group G is locally projectively solvable or an RN^*-group, then G is solvable. If G is locally projectively nilpotent, a nil group or an N-group (in the topological sense), then G is nilpotent.*

Obviously, locally projectively nilpotent (solvable) groups are a topological generalization of locally nilpotent (solvable) abstract groups. The structure of locally nilpotent (in the abstract sense) topological groups was established by V. M. Glushkov ([1]). It turns out that all the main results of ([1]) carry over to the case of locally projectively nilpotent groups. In this case the proofs of the main theorems from ([1]) are changed only insignificantly and rely additionally on the general Cartan–Mal' cev–Iwasawa theorem ([9]) and elementary properties of projective limits.

Let us formulate the main structural result, generalizing Theorems 8.3 and 8.5 of ([1]).

Theorem 9*. *Let G be a locally projectively nilpotent group, and let G_0 be the connected component of its identity.*

Then the following assertions hold: 1) the set B of all bicompact elements of the group G forms a closed invariant subgroup, and $B \subset Z_G(G_0)$; 2) the factor group G/B is a pure locally nilpotent Lie group; 3) G_0 is a nilpotent group; 4) the normal divisor $N = B \cdot G_0$ is open in G, and the factor group G/N is a discrete locally nilpotent torsion-free group; 5) the group N is isomorphic to the factor group $L \times A/C$, where L is a connected simply connected nilpotent Lie group; $A \cong B$; C is such a closed central subgroup of the direct product $L \times A$ that $C \cap A = (e)$, while $C \cap L$ is a discrete group.

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* *Proof-correction note.* As has become known to us, Theorem 9 was independently obtained also by V. I. Ushakov.