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Abstract

Full Text

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On the Theory of Integral Equations of Volterra Type

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1°. Let D be an arbitrary open set in n-dimensional space ($n \geqslant 1$). Consider the Fredholm integral equation of the second kind

$$y(P) = \lambda \int_D K(P,Q) y(Q)\, dQ + f(P), \qquad P \in D, \tag{1}$$

where

$$f(P) \in L_2(D), \qquad K(P,Q) \in L_2(D \times D).$$

As is known, if the kernel $K(P,Q)$ has no eigenvalues, then for any $\lambda \neq \infty$ there exists a unique solution $y(P) \in L_2(D)$ of equation (1), and it can be obtained by the method of successive approximations.

In this connection, criteria ensuring the absence of eigenvalues for kernels are of special interest. The most general such criteria are satisfied by kernels of the widely known equations of Volterra type. These equations, as a rule, are considered in the one-dimensional case ($n = 1$), when D is an interval (more rarely for $n \geqslant 2$, when D is an n-dimensional rectangle).

In the present note a generalization of the concept of an equation of Volterra type is introduced, making it possible, for an arbitrary n-dimensional set D, to distinguish classes of kernels having no eigenvalues.

2°. Definitions.

1. Let $S \subset D \times D$. A function $K(P,Q)$ will be called an S-kernel if $K(P,Q) \in L_2(D \times D)$ and $K(P,Q) = 0$ for $(P,Q) \notin S$.
2. An open set $S \subset D \times D$ will be called a **set of type** V if every S-kernel has no eigenvalues.
3. If $S \subset D \times D$ and $S_1 = D \times D - \overline{S}$ ($\overline{S}$ is the closure of S) are both sets of type V, then S will be called a **maximal set of type** V.
4. Let $S \subset D \times D$. We shall agree to write $P_1 \xrightarrow{S} P_2$ ($P_1 \xrightarrow{\overline{S}} P_2$) if $(P_1, P_2) \in S$ ($(P_1, P_2) \in \overline{S}$).
5. If S is a set of type V and $K(P,Q)$ is an S-kernel, then equation (1) will be called a **generalized equation of Volterra type**.

3°. Below a complete characterization of domains of type V is given.

Theorem 1. *In order that S be a set of type V, it is necessary and sufficient that, for every $k \geqslant 1$, from the conditions*

$$P_1 \xrightarrow{S} P_2 \xrightarrow{S} P_3 \xrightarrow{S} \cdots \xrightarrow{S} P_k \tag{2}$$

it follow that

$$P_k \xleftarrow{S} P_1. \tag{3}$$

Proof. a) **Sufficiency** will be proved if we show that the Fredholm determinant *

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \underbrace{\int_D \cdots \int_D}_{m} K(\xi_1, \dots, \xi_m)\, d\xi_1 \dots, d\xi_m, \tag{4}$$

* Formula (4) is valid if $K(P, Q) \in L_2(D \times D)$ (1).

where

$$K(\xi_1, \xi_2, \dots, \xi_m) = \det\{K(\xi_i, \xi_j)\} \quad (i, j = 1, 2, \dots, m) \tag{5}$$

has no zeros, except only where $K(P, Q) = 0$ outside the region S satisfying the conditions of the theorem. The general term in the expansion of the determinant (5), up to sign, has the form

$$K(\xi_1, \xi_{\alpha_1}) K(\xi_2, \xi_{\alpha_2}) \cdots K(\xi_m, \xi_{\alpha_m}), \tag{6}$$

where

$$\begin{pmatrix} 1 & 2 & \dots & m \\ \alpha_1 & \alpha_2 & \dots & \alpha_m \end{pmatrix}$$

is some permutation of m elements.

Since every permutation can be decomposed into cycles, the product (6) contains at least one group of factors of the form

$$K(\eta_1, \eta_2) K(\eta_2, \eta_3) \cdots K(\eta_k, \eta_1) \quad (1 \leq k \leq m). \tag{7}$$

The product (7) is equal to zero, since otherwise we would have

$$\eta_1 \xrightarrow{S} \eta_2 \xrightarrow{S} \eta_3 \xrightarrow{S} \cdots \xrightarrow{S} \eta_k \xrightarrow{S} \eta_1,$$

which contradicts conditions (2)–(3) of the theorem.

Thus all determinants (5) are equal to zero, i.e. $D(\lambda) \equiv 1$.

b) **Necessity.** The case $k = 1$ is trivial, since if $P_1 \xrightarrow{S} P_1$ (i.e. $(P_1, P_1) \in S$), then it is easy to construct a symmetric S-kernel $K(P, Q) \not\equiv 0$, having, as is known, at least one eigenvalue.

Now let $P_1, P_2, \dots, P_k \in D$ and

$$P_1 \xrightarrow{S} P_2 \xrightarrow{S} P_3 \xrightarrow{S} \cdots \xrightarrow{S} P_k \xrightarrow{S} P_1 \quad (k \geq 2). \tag{8}$$

We may assume that $P_i \neq P_j$ for $i \neq j$ $(i, j = 1, 2, \dots, k)$, since otherwise in the chain (8) the number k could be reduced.

Choose $\varepsilon > 0$ so small that the regions $\sigma_i \times \sigma_{i+1}$ $(i = 1, 2, \dots, k)$, where $\sigma_i \subset D$ are spheres* of radius ε with centers at the points P_i, respectively, lie in S and are pairwise disjoint.

Adopting the notation $\sigma_{i+k} = \sigma_i$, we may assume $i \geq 1$.

Now construct the S-kernel

$$K^*(P, Q) = \begin{cases} 1, & \text{if } P \in \sigma_i,\ Q \in \sigma_{i+1}\ (i \geq 1), \\ 0, & \text{in all other cases.} \end{cases} \tag{9}$$

Then the equation

$$y(P) = \lambda \int_D K^*(P, Q) y(Q)\, dQ, \qquad P \in D, \tag{10}$$

for $\lambda = \dfrac{1}{\omega}$, where $\omega = \operatorname{mes} \sigma_i$, has the nonzero solution

$$y(P) = \begin{cases} 1, & \text{if } P \in \sigma_i, \\ 0, & \text{if } P \notin \sigma_i, \end{cases} \quad (i \geq 1).$$

Indeed, for $P \notin \sigma_i$, from (9) and (10) it follows that $y(P) \equiv 0$, while for $P \in \sigma_i$ $(i \geq 1)$

$$y(P) = \lambda \int_{\sigma_{i+1}} y(Q)\, dQ.$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

* For $n = 1$, the σ_i are intervals, and for $n = 2$, circles.

4°. In order to characterize maximal sets of type V, for any $S \subset D \times D$ we shall assume $P \overset{S}{\leftarrow} P,\ P \in D$ (i.e. $(P, P) \in S$). This assumption does not diminish generality, since the values of the kernel $K(P, Q)$ on an n-dimensional manifold play no role.

Theorem 2. *In order that S be a maximal set of type V, it is necessary and sufficient that the following conditions hold:*

A. *If $P_1 \overset{S}{\leftarrow} P_2$ and $P_1 \neq P_2$, then $P_2 \overset{S}{\rightarrow} P_1$.*

B. *If $P_1 \overset{S}{\rightarrow} P_2 \overset{S}{\rightarrow} P_3$, then $P_1 \overset{S}{\rightarrow} P_3$.*

Fig. 1

Proof. a) **Necessity.** Let S be a maximal set of type V. Denote $S_1 = D \times D - S$; from Theorem 1 we obtain that if $P_1 \overset{S_1}{\rightarrow} P_2$, then $P_2 \overset{S_1}{\leftarrow} P_1$, i.e. condition A is satisfied. Condition B will be a simple consequence of condition (2) of Theorem 1 (for $k = 3$) and condition A.

b) **Sufficiency.** If the conditions of the theorem are fulfilled, then S is a set of type V. Indeed, from condition B it follows that if $P_1 \overset{S}{\rightarrow} P_2$, then $P_2 \overset{S}{\leftarrow} P_1$ (otherwise we would have $P_1 \overset{S}{\rightarrow} P_1$, which we have excluded). This observation permits, by induction from condition B, obtaining all the conditions of Theorem 1.

Applying now condition A, in exactly the same way we prove that $S_1 = D \times D - S$ is also a set of type V, which was required to be proved.

Fig. 2

Let us note the following consequences:

1. If S is a maximal set of type V, then any proper extension $S_1 \supset S$ $(S_1 \neq S)$ is no longer a set of type V.
2. If S is a maximal set of type V, and $K_1(P, Q)$ and $K_2(P, Q)$ are S-kernels, then their composition

$$\Omega(P,Q) = \int_D K_1(P,R)K_2(R,Q)\,dR$$

is also an S-kernel. (If S is a set of type V, but not maximal, this is, generally speaking, false.)

3. There is a definite correspondence between maximal sets of type V and ways of ordering the points of the set D. The well-known one-dimensional equation of Volterra type (see below (11)) corresponds to the case where D is an interval, and the symbol $\overset{S}{\to}$ coincides with the usual inequality symbol $>$.

5°. As an illustration, let us consider the simplest one-dimensional case, where D is the interval $(0,1)$. In Fig. 1 the simplest maximal sets of type V are shaded. Of these sets, the first (a triangle) corresponds to the classical Volterra equation

$$y(x) = \lambda \int_0^x K(x,t)y(t)\,dt + f(x) \qquad (0 < x < 1). \tag{11}$$

In Fig. 2 the simplest sets that are not sets of type V are shaded. (The trivial case, when the set contains points of the line $x = t$, is not shown.)

6°. Using the results given above, it is not difficult to prove that if

$$\sum_{i=1}^{\infty} \int_D |f_i(P)|^2\,dP < +\infty, \tag{12}$$

$$\sum_{i,j=1}^{\infty} \int_D \int_D |K_{ij}(P,Q)|^2\,dP\,dQ < +\infty \tag{13}$$

and $K_{ij}(P,Q)$ $(i,j \geq 1)$ are S-kernels, with S a set of type V, then the system

$$y_i(P) = \sum_{j=1}^{\infty} \int_D K_{ij}(P,Q)y_j(Q)\,dQ + f_i(P) \qquad (i \geq 1)$$

has a unique solution $\{y_i(P)\}_1^\infty$, satisfying condition (12).

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Note: Figure translations are in progress. See original paper for figures.

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