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Abstract

Full Text

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PROOF OF ANALOGUES OF THE MINKOWSKI AND VORONOI THEOREMS FOR THREE-DIMENSIONAL ISOGONAL DECOMPOSITIONS ASSOCIATED WITH A GROUP OF PARALLEL TRANSLATIONS

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A decomposition of space is called **normal** if its bodies are convex polyhedra and are adjacent along entire faces. On the plane and on the sphere, the most general normal decompositions—both isohedral* and isogonal**—have been considered. In Euclidean space of three or more dimensions, up to now only isohedral decompositions for the group of parallel translations (parallelohedra) have been considered. Isohedral decompositions for an arbitrary group (stereohedra) were first considered in the paper ([1]).

In the present note, general normal isogonal decompositions of three-dimensional Euclidean space associated with a group of parallel translations are considered for the first time.

If a decomposition $\{Q\}$ is normal and isogonal with respect to a group of parallel translations T, then it satisfies the following conditions:
1) if O is any vertex of the decomposition, then the totality of all vertices of the decomposition forms an exact lattice $\Gamma = \{O\tau\}$;
2) under any translations T the decomposition is transformed into itself.

Theorem 1. *The decomposition $\{Q\}$ is symmetric with respect to any point of the lattice Γ and to the midpoint of any segment joining two points of Γ.*

Proof. Reflecting the decomposition $\{Q\}$ in the point O, we obtain another, likewise isogonal, decomposition $\{Q^*\}$, the totality of whose vertices is the same lattice Γ.

Let ω and ω^* be the totalities of all those edges of the decompositions $\{Q\}$ and, respectively, $\{Q^*\}$, which issue from the point O, and suppose that in ω there is an edge OA ($\overrightarrow{OA}$ is some vector $\vec{t}$ from T). Translating $\{Q\}$ by the vector $-\vec{t}$, we move the point A to the point O, and the point O to the point A^*, symmetric to A with respect to the point O. Therefore each vector in the totality ω has in ω a vector symmetric to it with respect to the point O; i.e., the totality ω is symmetric with respect to the point O, and, consequently, $\omega = \omega^*$. From the fact that the totality of all edges of the decomposition $\{Q\}$ is $\{\omega\tau\}$, it follows that the totalities $\{\omega\tau\}$ and $\{\omega^*\tau\}$ of edges of the decompositions $\{Q\}$ and $\{Q^*\}$ coincide.

Suppose $\{Q\} \neq \{Q^*\}$. Then there are in $\{Q\}$ a polyhedron Q, and in $\{Q^*\}$ a polyhedron Q^*, such that $Q \neq Q^*$ and $Q \cap Q^* = D$, where D is some three-dimensional polyhedron. In view of $Q \neq Q^*$, it cannot simultaneously be the case that $D = Q$ and $D = Q^*$. Let $D \neq Q$. Then there exist points C and M such that C is an interior point of the polyhedron D, and $M \notin D$, but $M \in Q$. The segment CM intersects the boundary of the polyhedron D at a certain point F, which is an interior point of this segment. If the point F lies on an edge of the polyhedron D, then the point C can be moved slightly so that F will be an interior point of some face E of the polyhedron D; $E \subset P^*$, where P^* is some face of the polyhedron Q^*. Denote by S the intersection of the plane of the face P^* with the polyhedron Q. It is obvious that $E = S \cap P^*$

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* Those for which the group of coincidences is transitive with respect to the bodies of the decomposition.
** Those for which the group of coincidences is transitive with respect to the vertices of the decomposition.

The point F is an interior point also for S, since it is an interior point of the polyhedron Q. From the fact that the point F is interior both for E and for S, it follows that E is a convex polygon which either 1) is contained in S, or 2) coincides with S.

In case 1), there will be points C_1 and M_1 such that C_1 is an interior point of the polygon E, while $M_1 \notin E$, but $M_1 \in S$, and such that the segment C_1M_1 intersects some side E_1 of the face E at a point F_1, which is an interior point of E_1 and of the segment C_1M_1. Consequently, the point F_1 of the edge is at the same time an interior point of S, i.e., of the polyhedron Q, which is impossible.

In the second case, in the polyhedron Q there will be two vertices B_1 and B_2 such that they lie on different sides of the plane S, since S is not a face of the polyhedron Q.

In view of the convexity of the polyhedron Q, the segment B_1B_2 intersects the face E, and consequently also the face P^*, at some point F_2, which is not a vertex of Q.

Consider three cases.

A. The point F_2 is an interior point of an edge. This is impossible, since it is simultaneously an interior point of the face P^*.

B. The point F_2 is an interior point of some face P of the polyhedron Q. Reflect the partition $\{Q\}$ in the midpoint of the segment B_1B_2. In view of the fact that this reflection is equivalent to reflection in the point B_1 of the lattice Γ and then to translation by the vector $\overrightarrow{B_1B_2}$ of the lattice Γ, the partition $\{Q\}$ will pass into the partition $\{Q^*\}$, the polyhedron Q into some polyhedron Q_1^*, and the face P will pass into some face P_1^* of the polyhedron Q_1^*. All interior points of the segment B_1B_2 were interior points of the face P and, consequently, will pass into interior points of the face P_1^*. The point F_2, consequently, will be simultaneously an interior point both of the face P^* and of the face P_1^*; but this is impossible, since the faces P^* and P_1^* do not coincide, in view of the fact that the vertices B_1 and B_2 do not belong to the face P^*.

C. The point F_2 is an interior point of the polyhedron Q. Having made the reflection in the midpoint of the segment B_1B_2, we shall analogously see that F_2 will be simultaneously an interior point of the polyhedron Q_1^* and a point of the face P^*, which is impossible. The theorem is thus proved.

Theorem 2. *Every partition $\{Q\}$ can be transformed by an affine transformation into a partition $\{L\}$ ($\{L\}$ in the sense of Voronoi).*

Proof. We divide all polyhedra $\{Q\}$ into equivalence classes: in one class we put those polyhedra which are obtained from one another by translations of the group T.

Lemma. *There exist such polyhedra of the partition $\{Q\}$, one taken from each equivalence class, that in sum they form some fundamental parallelepiped of the lattice Γ.*

Let us prove the lemma. Let P be any face of the partition $\{Q\}$, and let Π be its plane.

1. Then the whole plane Π is composed of faces of the partition $\{Q\}$, since to each face lying in the plane Π one can attach an adjacent face along any of its sides, also lying in this plane, by reflection in the midpoint of this side.

2. Let BB_1 be an edge of the polyhedron Q, issuing from the vertex B of its face P and not belonging to this face. Translating the plane Π of the face P by the vector $\overrightarrow{BB_1}$, we obtain a plane Π', parallel to the plane Π. In the region Φ enclosed between the planes Π and Π', there are no points of the lattice Γ.

Suppose that in the region Φ there were a point B_2 of the lattice Γ. Then, translating the plane Π by the vector $\overrightarrow{BB_2}$, we would obtain some plane Π_1, which lies inside the region Φ and, consequently, intersects the edge BB_1 at one of its interior points, which is impossible in view of 1.

Consider, besides the face P, two more faces P' and P'' of this polyhedron.

face Q, issuing from the vertex B. For each of them we construct, analogously, the domains Φ', Φ''. The intersection $\overline{\Phi} \cap \overline{\Phi'} \cap \overline{\Phi''}$ of the closures of all these three domains is a parallelepiped which, in view of the property of the domains Φ, Φ', Φ'', has inside itself, inside its faces, and inside its edges no points of Γ. From consideration of the intersection of the planes of the faces of the tiling $\{Q\}$ and of the emptiness of the domains Φ, Φ', Φ'', it follows that the edges of the parallelepiped will be edges of the tiling $\{Q\}$, and its vertices will be vertices of the tiling $\{Q\}$, i.e., points of Γ. Consequently, the parallelepiped obtained is a fundamental one in the lattice Γ. The lemma is thus proved.

In order to find all possible tilings (we shall not regard affine images of one another as different), it is enough to find all possible tilings of a parallelepiped into polyhedra Q' satisfying the following conditions:

1. The vertices of the polyhedra Q' are the vertices of the parallelepiped, and only these.
2. The polyhedra Q' are adjacent along whole faces.
3. The tiling $\{Q'\}$ of the parallelepiped is symmetric with respect to its center C.

Let us consider three cases.

A. The center C lies inside one of the polyhedra Q'_1 of the tiling $\{Q'\}$. Then in the polyhedron Q'_1, for each of its vertices there is a vertex symmetric with respect to the center C. In view of the fact that centrally symmetric vertices are the ends of a space diagonal of the parallelepiped, the polyhedron Q'_1 has as vertices the ends of either three such diagonals or four. In the first case Q'_1 is an octahedron, and the remaining parts of the parallelepiped are two tetrahedra. In the second case Q'_1 is the parallelepiped itself.

B. The center C lies inside one of the faces P of the tiling $\{Q'\}$. Arguing analogously, we see that the vertices of this face P are the ends of two space diagonals. The face P is a diagonal plane of the parallelepiped and cuts it into two triangular prisms symmetric to one another.

Only two cases are possible:

1. These prisms themselves form a tiling $\{Q'\}$ of the parallelepiped.
2. If from one of them one cuts off a tetrahedron, and from the other the tetrahedron symmetric to it, then the parallelepiped is divided into two quadrangular pyramids and two tetrahedra.

C. The center C is an interior point of some edge l of the tiling $\{Q'\}$. It is obvious that the edge l, by virtue of the symmetry of the tiling $\{Q'\}$, is a space diagonal of the parallelepiped. At the edge l no fewer than three faces meet; but in the present case each such face has a face symmetric to it with respect to the midpoint of the edge l, and therefore four or six faces meet at the edge l. In the first case the parallelepiped is divided into two tetrahedra and two quadrangular pyramids. Each quadrangular pyramid, in turn, can be divided into two tetrahedra by drawing certain diagonals of their quadrangular faces. If the other diagonals of their bases are drawn, a different tiling of the parallelepiped into six tetrahedra is obtained.

The seven cases of tilings $\{Q'\}$ considered exhaust all possible cases of the required tilings of the parallelepiped. Taking this into account, it is easy to see that all tilings $\{Q\}$ are affinely equivalent to the five known tilings $\{L\}$.

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1. B. N. Delone, N. N. Sandakova. *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **64**, 28 (1961).

Note: Figure translations are in progress. See original paper for figures.