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Abstract

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THE MIXED PROBLEM FOR GENERAL PARABOLIC SYSTEMS IN A HALF-SPACE

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1. The present note is devoted to the study of the mixed problem for a system parabolic in the sense of I. G. Petrovskii (^{1}), containing derivatives with respect to t of order higher than the first, with constant coefficients in a half-space. The extension of the results obtained to the case of the mixed problem in a finite part of space and to the case of systems with variable coefficients can be carried out in the same way as was done in our work (^{2}), by using half-space potentials and reducing, with their aid, the mixed problem to resolvable integral equations.
2. Let

$$\mathfrak{A}u = \left[A_0\left(\frac{\partial}{\partial t}\right) - A_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)\right] u = 0; \tag{1}$$

$$\left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=+0} = 0 \qquad (k = 0, 1, \dots, n_i - 1;\ i = 1, 2, \dots, N); \tag{2}$$

$$\lim_{x_n \to 0} B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u(x,t) = f(x', t), \qquad x_n > 0, \quad |x_j| \leq \infty\ (j \leq n-1), \tag{3}$$

where the row operator matrix of width N is

$$A_0\left(\frac{\partial}{\partial t}\right) = \left[\frac{\partial^{n_1}}{\partial t^{n_1}}, \dots, \frac{\partial^{n_N}}{\partial t^{n_N}}\right];$$

$$A_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \left\| \sum_{(k)} A_{ij}^{k_0, k_1, \dots, k_n} \frac{\partial^{k_0 + k_1 + \dots + k_n}}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_n^{k_n}} \right\|_{i,j=1}^{i,j=N};$$

$$B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \left\| \sum_{((k))_{ij}} B_{ij}^{k_0, \dots, k_n} \frac{\partial^{k_0 + \dots + k_n}}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_n^{k_n}} \right\|_{i,j=1}^{i=aM,\ j=N};$$

$M = n_1 + \dots + n_N$; $A_{ij}^{k_0,\dots,k_n}$, $B_{ij}^{k_0,\dots,k_n}$ are certain constants; $2a$ is the so-called parabolic weight; (k) denotes summation over all systems of numbers $k_0, k_1, \dots, k_n$ for which $2ak_0 + k_1 + \dots + k_n = 2an_i$; $((k))_{ij}$ is likewise summation over all systems of numbers $k_0, k_1, \dots, k_n$ for which $2ak_0 + k_1 + \dots + k_n = 2an_j + m_i$; m_i are arbitrary integers such that $2an_j + m_i \geq 0$.

It is assumed that system (1) is parabolic, and that condition (α) of regular solvability of problem (1), (2), (3) is fulfilled, i.e. the rank of the matrix $\mathfrak{M}$ is equal to aM, where

$$\mathfrak{M} = \int_{(\Gamma^+)} B(p, i\alpha)\, \mathfrak{A}^{-1}(\alpha, p)\, T(\alpha_n)\, d\alpha_n, \qquad T(\alpha_n) = \left\|\mathcal{E},\ \alpha_n\mathcal{E}, \dots, \alpha_n^{2aM-1}\mathcal{E}\right\|;$$

$\mathcal{E}$ is the identity matrix of size $N \times N$; $\operatorname{Re} p \geq -\delta_1|\alpha'|^{2a}$, $\delta > \delta_1 > 0$, δ is the number entering into the definition of parabolicity; (Γ^+) is a simple con-

contour in the complex a_n-plane, inside which lie all aM roots $a_n^{(k)}$ of the equation $\det \mathfrak{A}(a, p) = 0$ with positive imaginary part. It is required to find a solution of system (1) in the domain

$$D\ (t > 0,\ x_n > 0,\ |x_j| < \infty), \qquad j \leq n - 1,$$

satisfying conditions (2), (3).

Theorem 1. *Problem (1), (2), (3), under condition (a), has a solution representable by the formula*

$$u = \int_0^t d\tau \int_{-\infty}^{\infty} G(x' - \xi', x_n, t - \tau) f(\xi', \tau)\, d\xi', \tag{4}$$

$$G(x', x_n, t) = \frac{1}{(2\pi)^{ni}} \int_{-\infty}^{\infty} d\alpha' \int_{\sigma - i\infty}^{\sigma + i\infty} e^{i(\alpha' x') + pt} \int_{(\Gamma^+)} e^{i\alpha_{nx}n} \mathfrak{A}^{-1}(\alpha, p) Z(a_n)\, d\alpha_n\, R^{-1}(\alpha', p).$$

The kernel G of the solution has an estimate of the form

$$\left|\frac{\partial^{m+l} G_{ij}(x', x_n, t)}{\partial t^m \partial x_1^{k_1} \cdots \partial x_n^{k_n}}\right| \leq \frac{C_1 \exp\{-C_2[\,|x'|^{2a/(2a-1)} + x_n^{2a/(2a-1)}\,]t^{-1/2a}\}}{t^{(2am+2a+n-m_j-2an_i+l-1)(2a)^{-1}}}, \tag{5}$$

$$l = k_1 + \dots + k_n.$$

For $(BG)_{ij}$ we obtain the estimate

$$|BG|_{ij} \leq \frac{C_1 x_n \exp\{-C_2[\,|x'|^{2a/(2a-1)} + x_n^{2a/(2a-2)}\,]t^{-1/(2a-1)}\}}{t^{(2a+n+m_i-m_j)(2a)^{-1}}}, \tag{6}$$

$Z(a_n)$ is a matrix of dimensions $N \times aM$, chosen so that

$$\det \int_{(\Gamma^+)} B(p, i\alpha)\mathfrak{A}^{-1}(\alpha, p) Z(a_n)\, d\alpha_n \neq 0,$$

$$R(\alpha', p) = \int_{(\Gamma^+)} B(p, i\alpha)\mathfrak{A}^{-1}(\alpha, p) Z(a_n)\, d\alpha_n.$$

Let $\bar{m} = \max m_i,\ \underline{m} = \min m_i,\ \bar{m} - \underline{m} = 2ak + l,\ 0 \leq l < 2a$. If $\bar{m} = \underline{m}$, then $f_j(x', t)$ is continuous and is subject to the condition

$$|f_j(x', t)| \leq Ce^{\sigma t}, \qquad C, \sigma > 0. \tag{7}$$

If $\bar{m} \neq \underline{m}$, then it is additionally assumed that

$$\left|\frac{\partial^r f_j}{\partial t^r}\right| \leq Ce^{\sigma t}, \qquad \frac{\partial^r f_j(x', 0)}{\partial t^r} = 0, \qquad r = 0, 1, \dots, k+1, \tag{8}$$

$\partial^r f_j / \partial t^r$ are continuous functions of their arguments, $j \leq aM$.

3. The method of solving problem (1), (2), (3) is a generalization of the method developed by us earlier (^{3}).

Let us note, however, the following features of problem (1), (2), (3), which make this generalization nontrivial:

1) In the previous problem, the initial case was the case of equal orders in all rows of the boundary operator. In the problem under consideration, the case of equal orders in the rows, as well as the case of the first boundary-value problem, leads to high singularities in the kernel, and only a special choice of boundary conditions makes it possible to arrive at a kernel with an admissible singularity.

2) A distinctive formulation of the condition of regular solvability.

3) Extension of the results to the case of boundary conditions with arbitrary order of derivatives and the associated deep compatibility of the boundary and initial conditions.

Applying the Fourier transform in $x_1, \dots, x_{n-1}$ and the Laplace transform in t to the problem (1), (2), (3), we arrive at a boundary-value problem for a system of ordinary equations of the form

$$A_0\,\bar{u}(p) - A_1\left(p, ia', \frac{d}{dx_n}\right)\bar{u} = 0; \tag{9}$$

$$\lim_{x_n\to 0} B\left(p, ia', \frac{d}{dx_n}\right)\bar{u} = \overline{f(a',p)}, \tag{10}$$

whose solution is written in the form

$$\bar{u} = \int_{(\Gamma^+)} e^{ia_\alpha x_n}\mathfrak{A}^{-1}(a,p)T(a_n)\,da_n\,C;$$

C is a column of arbitrary parameters of height $2aMN$.

Relying on conditions (α) and (10), we obtain

$$\bar{u} = \int_{(\Gamma^+)} e^{ia_n x_n}\mathfrak{A}^{-1}(a,p)Z(a_n)\,da_n\,R^{-1}(a',p)\,\overline{f(a',p)},$$

and, applying the inverse Fourier and Laplace transforms, we obtain the solution of the problem (1), (2), (3) in the form (4). In justifying formula (4), the estimates (5) and (6) are of fundamental importance.

The integral over the contour (Γ^+) is an analytic function of a_j $(j \leq n-1)$ under the condition $|p| \geq \varepsilon > 0$, at least in some strip adjacent to the real axis.

It is proved that one can vary the contour of integration with respect to p so that the integral has meaning also for $x_n = 0$, and so that on the contour $|p| \geq \varepsilon > 0$. In the integral $G(x', x_n, t)$, under the integral sign with respect to a', there is a function analytic in the strip of a_j. Using the permissible shift of all real lines of integration with respect to a_j into the complex plane of a_j, we obtain the estimate (5) of the kernel G.

Let $\bar{m} = \bar{m}$. For this case, using (6), one may write

$$BC = \frac{\exp\left\{-C_1\left[|x'-\xi'|^{2a/(2a-1)} + x_n^{2a/(2a-1)}\right](t-\tau)^{-1/(2a-1)}\right\}x_n H\left(\dfrac{x'-\xi'}{(t-\tau)^{1/2a}}, \dfrac{x_n}{(t-\tau)^{1/2a}}\right)}{(t-\tau)^{(2a+n)/2a}}. \tag{11}$$

The bounded function $H(y_1, y_n)$ tends to zero as $y_n \to 0$,

$$B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u(x,t) = \int_0^t d\tau\int_{-\infty}^{\infty} B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)G(x'-\xi', x_n, t-\tau)\,f(\xi',\tau)\,d\xi'.$$

Substituting into Bu the expression BG from (11) and making the change of variables

$$\frac{x_j - \xi_j}{(t-\tau)^{1/2a}} = \lambda_j, \qquad \frac{x_n}{(t-\tau)^{1/2a}} = \lambda_n, \qquad j = 1, 2, \dots, n-1,$$

we obtain

$$Bu = \int_{x_n t^{-1/2a}}^{\infty} d\lambda_n \int_{-\infty}^{\infty} \exp\left\{-C_1 \left[|\lambda'|^{2a/(2a-1)} + \lambda_n^{2a/(2a-1)}\right]\right\} H(\lambda', \lambda_n) f\left(x' - \lambda' \frac{x_n}{\lambda_n^{2a}}, t - \frac{x_n^{2a}}{\lambda_n^{2a}}\right) d\lambda'.$$

Using (7), we note that here passage to the limit under the integral sign as $x_n \to 0$ is legitimate, and

$$\lim_{x_n \to 0} Bu = K f(x', t),$$

where K is a constant independent of the form of the function $f(x', t)$.

Assuming that $f(x', t)$ is a finite function, we note that the integral Bu admits passage to the limit as $x_n \to 0$, and we directly obtain

$$\lim_{x_n \to 0} Bu = f(x', t), \qquad \text{i.e. } K = \mathcal{E}.$$

For $\overline{m} \neq \widetilde{m}$, integrating by parts with respect to τ and using conditions (8), we again arrive at integrals that allow passage to the limit under the integral sign, and again

$$\lim_{x_n \to 0} Bu = f(x', t).$$

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CITED LITERATURE

[1] I. G. Petrovskii, *Bull. Moscow Univ.*, Ser. A, **7** (1937).
[2] T. Ya. Zagorskii, *Mixed Problems for Parabolic Systems*, Lvov, 1961.

Note: Figure translations are in progress. See original paper for figures.

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