CYBERNETICS AND CONTROL THEORY

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Abstract

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ON THE STABILITY OF CONTROLLED PROCESSES

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In this article we consider the question of the stability of a controlled process defined by a system of differential equations

$$\frac{df_i}{dt} = F_i(f_1, f_2, \dots, f_n, t, u), \qquad i = 1, 2, \dots, n, \tag{1}$$

where $f_1, f_2, \dots, f_n$ are the controlled functions, for which a prescribed law of their variation $f_i = f_i^{(0)}(t)$ must be ensured, and $u(t)$ is the control function. It is assumed that the input signal of the automatic control system $v(t)$ is formed as a linear combination of the mismatches $f_i - f_i^{(0)}$, and that control is carried out by some linear system with variable parameters (systems of automatic control of this kind are the subject of A. V. Solodov’ s monograph (1)).

In the linear formulation, the problem of stability of the controlled process reduces to the study of the equations

$$\frac{dx}{dt} = A(t)x + a(t)u, \qquad v = (b(t), x), \tag{2}$$

where $x(t)$ is an n-dimensional vector with components $f_i - f_i^{(0)}$, $A(t)$ is the matrix composed of the elements

$$a_{ij} = \left(\frac{\partial F_i}{\partial f_j}\right)_{f_1 = f_1^{(0)}, \dots, f_n = f_n^{(0)},\, u=0},$$

$a(t)$ is an n-dimensional vector whose components are respectively equal to the derivatives $\partial F_i / \partial u$, computed for $f_1 = f_1^{(0)}, \dots, f_n = f_n^{(0)}$, $u = 0$, and $b(t)$ is an n-dimensional vector determining the law of formation of the input signal of the control system. Equations (2) must be considered together with the equation determining the relation between the control function $u(t)$ and the input signal of the control system $v(t)$.

We shall proceed from the case when $u(t) = 0$ for $t < t'$, $v(t) = 0$ for $t < t'$ and $t > t' + \varepsilon$, and $v(t) = 1/\varepsilon$ for $t' < t < t' + \varepsilon$. Let in this case $u(t) = r_\varepsilon(t - t', t')$ for $t > t'$. By virtue of the assumed linearity of the automatic control system, the dependence between the control function $u(t)$ and the input signal $v(t)$ can be expressed by the formula

$$u(t) = \int_{-\infty}^{t} v(t')\, r(t - t', t')\, dt', \tag{3}$$

where $r(t - t', t') = \lim_{\varepsilon \to 0} r_\varepsilon(t - t', t')$.

Consider a control system with constant parameters whose values coincide with the values of the parameters of the automatic control system under study at $t = t_0$. For $v(t) = e^{\lambda t}$ for

for the function $u(t)$ we obtain the expression

$$u(t) = \int_{-\infty}^{t} r(t - t', t_0) e^{\lambda t'}\, dt' = \int_{0}^{\infty} r(s, t_0) e^{\lambda (t-s)}\, ds = R(\lambda, t_0) v(t), \tag{4}$$

where

$$R(\lambda, t) = \int_{0}^{\infty} r(s, t) e^{-\lambda s}\, ds \tag{5}$$

is the transfer function of the automatic-control system under study with parameters "frozen" at the given instant of time t.

If, for $t = t_0$, along with the parameters of the control system one also "freezes" the parameters of the controlled object, the investigation of stability reduces to consideration of the equations

$$\frac{dx}{dt} = A(t_0)x + a(t_0)u, \qquad v = (b(t_0), x). \tag{6}$$

together with equation (4). For $u = e^{\lambda t}$, equations (6) will have the solution

$$x = [\lambda E - A(t_0)]^{-1} a(t_0) u, \qquad v = R_0(\lambda, t_0) u, \tag{7}$$

where $R_0(\lambda, t_0) = (b(t_0), [\lambda E - A(t_0)]^{-1} a(t_0))$ is the transfer function of the controlled object. Comparing (4) and (7), we obtain the well-known equation of automatic-control theory

$$R_0(\lambda, t_0) R(\lambda, t_0) = 1, \tag{8}$$

to which the method of "freezing" the parameters of the controlled object and the control system leads.

We indicate below a method by means of which one can investigate the stability of a controlled process under a slow variation of the parameters of the controlled object and the control system, without resorting to "freezing" these parameters. In this method we use the concept of slow time, introduced into mathematical physics by N. N. Bogolyubov and N. M. Krylov (^{2}).

In what follows we shall assume that the processes corresponding to solutions of equation (8) close to the poles of the transfer function $R(\lambda, t_0)$ are certainly stable, and that the stability of the controlled process is determined by the character of the solutions of the system of equations (2) and (3) that are close in their behavior to the solutions of the "open-loop" system $dx/dt = A(t)x$. To study these solutions we shall consider, instead of equations (2) and (3), the equations

$$\frac{dx}{dt} = A(\tau)x + \varepsilon a(\tau)u, \qquad v = (b(\tau), x),$$

$$u(t) = \int_{-\infty}^{t} v(t')r(t-t', \tau')\,dt', \tag{9}$$

where $\tau = \varepsilon t$ (respectively, $\tau' = \varepsilon t'$) is the slow time. Equations (9) become equations (2) and (3) for $\varepsilon = 1$. We shall seek a solution of equations (9) in the form

$$x(t) = y(\tau)\exp\theta(t), \qquad \theta'(t) = \lambda(\tau), \qquad \|y(\tau)\| = \sqrt{(y(\tau), \overline{y(\tau)})} = 1. \tag{10}$$

For $\varepsilon = 1$ the norm of the solution of equations (9) found in this way will satisfy the equation

$$\frac{d\|x(t)\|}{dt} = \operatorname{Re}\lambda(t)\,\|x(t)\|, \tag{11}$$

and as a criterion of local stability of the process corresponding to this solution of equations (9), the criterion

$$\operatorname{Re}\lambda(t) < 0. \tag{12}$$

may be adopted.

Substituting (10) into (9), we obtain the equation

$$\varepsilon y'(\tau) + \lambda(\tau) y(\tau) = A(\tau) y(\tau) + \varepsilon w(\tau) a(\tau), \tag{13}$$

where

$$w(\tau) = \int_{-\infty}^{t} (b(\tau'), y(\tau'))\, r(t - t', \tau') \exp[\theta(t') - \theta(t)]\, dt'$$
$$= \int_0^{\infty} c(\tau - \varepsilon s)\, r(s, \tau - \varepsilon s) \exp[\theta(t - s) - \theta(t)]\, ds,$$

$$c(\tau) = (b(\tau), y(\tau)). \tag{14}$$

Using the expansions

$$\begin{aligned}
c(\tau - \varepsilon s) &= c(\tau) - \varepsilon s c'(\tau) + \tfrac{1}{2}\varepsilon^2 s^2 c''(\tau) - \cdots,\\
r(s, \tau - \varepsilon s) &= r(s, \tau) - \varepsilon s r_\tau(s, \tau) + \tfrac{1}{2}\varepsilon^2 s^2 r_{\tau\tau}(s, \tau) - \cdots,\\
\exp[\theta(t - s) - \theta(t)] &= \exp[-s\lambda(\tau) + \tfrac{1}{2}\varepsilon s^2 \lambda'(\tau) - \tfrac{1}{6}\varepsilon^2 s^3 \lambda''(\tau) + \cdots]\\
&= \exp[-\lambda(\tau) s]\{1 + \tfrac{1}{2}\varepsilon s^2 \lambda'(\tau) + \varepsilon^2[\tfrac{1}{8} s^4 (\lambda'(\tau))^2 - \tfrac{1}{6} s^3 \lambda''(\tau)] + \cdots\},
\end{aligned}$$

we obtain for the function $w(\tau)$ the series

$$w(\tau) = w_0(\tau) + \varepsilon w_1(\tau) + \varepsilon^2 w_2(\tau) + \cdots, \tag{15}$$

where

$$w_0(\tau) = \int_0^{\infty} c(\tau)\, r(s, \tau) \exp[-\lambda(\tau) s]\, ds,$$

$$w_1(\tau) = \int_0^{\infty} [-s c'(\tau) r(s, \tau) - s c(\tau) r_\tau(s, \tau) +$$

$$+ \tfrac{1}{2} s^2 c(\tau) r(s, \tau) \lambda'(\tau)] \exp[-\lambda(\tau) s]\, ds, \ldots,$$

or, according to (5),

$$\begin{aligned}
w_0(\tau) &= c(\tau) R[\lambda(\tau), \tau],\\
w_1(\tau) &= c'(\tau) R_\lambda[\lambda(\tau), \tau] + c(\tau) R_{\lambda t}[\lambda(\tau), \tau] +\\
&\quad + \tfrac{1}{2} c(\tau) \lambda'(\tau) R_{\lambda\lambda}[\lambda(\tau), \tau], \ldots .
\end{aligned} \tag{16}$$

Denote by $\lambda_j(t)$, $j = 1, 2, \dots, n$, the eigenvalues of the matrix $A(t)$, and by $\xi_j(t)$ and $\eta_j(t)$, $j = 1, 2, \dots, n$, the eigenvectors of the matrices $A(t)$ and $A^*(t)$, satisfying the normalization conditions

$$\|\xi_j(t)\| = 1, \qquad (\xi_j(t), \eta_j(t)) = 1, \qquad j = 1, 2, \dots, n. \tag{17}$$

Generalizing to the system of integro-differential equations (13) the known method of asymptotic integration of ordinary differential equations, we shall seek the function $\lambda(\tau)$ and the vector $y(\tau)$ in the form of series

$$\begin{aligned} \lambda(\tau) &= \lambda_j(\tau) + \varepsilon\lambda_j^{(1)}(\tau) + \varepsilon^2\lambda_j^{(2)}(\tau) + \cdots, \\ y(\tau) &= \xi_j(\tau) + \varepsilon y_j^{(1)}(\tau) + \varepsilon^2 y_j^{(2)}(\tau) + \cdots. \end{aligned} \tag{18}$$

According to (15) and (16), for the function $w(\tau)$ we obtain the expansion

$$\begin{aligned} w = (b, \xi_j)R(\lambda_j, \tau) + \varepsilon\{(b, y_j^{(1)})R(\lambda_j, \tau) + [\lambda_j^{(1)}(b, \xi_j) + (b', \xi_j) \\ + (b, \xi_j')]R_\lambda(\lambda_j, \tau) + (b, \xi_j)[R_{\lambda t}(\lambda_j, \tau) + \tfrac{1}{2}\lambda_j' R_{\lambda\lambda}(\lambda_j, \tau)]\} + \cdots. \end{aligned} \tag{19}$$

Substituting the series (18) and (19) into equation (13) and comparing the coefficients of equal powers of ε in the left- and right-hand sides of the relation obtained, we get the equations

$$Ay_j^{(1)} = \lambda_j y_j^{(1)} + \lambda_j^{(1)}\xi_j + \xi_j' - (b, \xi_j)R(\lambda_j, \tau)a; \tag{20}$$

$$\begin{aligned} Ay_j^{(2)} = \lambda_j y_j^{(2)} + \lambda_j^{(2)}\xi_j + \lambda_j^{(1)} y_j^{(1)} + y_j^{(1)\prime} \\ - \{(b, y_j^{(1)})R(\lambda_j, \tau) + [\lambda_j^{(1)}(b, \xi_j) + (b', \xi_j) + (b, \xi_j)]R_\lambda(\lambda_j, \tau) \\ + (b, \xi_j)[R_{\lambda t}(\lambda_j, \tau) + \tfrac{1}{2}\lambda_j' R_{\lambda\lambda}(\lambda_j, \tau)]\}a \end{aligned} \tag{21}$$

and so on. The normalization condition imposed on the vector $y(\tau)$ in formulas (10), according to (17), will be fulfilled if the vectors $y_j^{(1)}(\tau), y_j^{(2)}(\tau), \dots$ satisfy the conditions

$$(y_j^{(1)}, \bar{\xi}_j) + (\bar{y}_j^{(1)}, \xi_j) = 0, \qquad (y_j^{(2)}, \bar{\xi}_j) + (\bar{y}_j^{(2)}, \xi_j) + (y_j^{(1)}, \bar{y}_j^{(1)}) = 0, \dots \tag{22}$$

We shall assume that, in the time interval under consideration, $\lambda_i(t) \neq \lambda_k(t)$ for $i \neq k$. In this case equation (20) for the vector y will be solvable when

$$\lambda_j^{(1)} = (a, \eta_j)(b, \xi_j)R(\lambda_j, \tau) - (\xi_j', \eta_j), \tag{23}$$

and the general solution of equation (20) will have the form

$$y_j^{(1)} = (b,\xi_j)R(\lambda_j,\tau)\sum_{k\neq j}\frac{(a,\eta_k)\xi_k}{\lambda_j-\lambda_k} - \sum_{k\neq j}\frac{(\xi_j',\eta_k)\xi_k}{\lambda_j-\lambda_k} + C_j^{(1)}\eta_j, \qquad (24)$$

where $C_j^{(1)}(\tau)$ is an arbitrary function; by a suitable choice of it one can satisfy the first of conditions (22). In an analogous way one can determine the function $\lambda_j^{(2)}(\tau)$ from the solvability condition for equation (21) and construct the vector $y_j^{(2)}(\tau)$, satisfying equation (21) and the second of conditions (22), and so on. Setting $\varepsilon = 1$ in (18), we obtain, according to (12), the stability criterion

$$\mathrm{Re}[\lambda_j(t) + \lambda_j^{(1)}(t) + \lambda_j^{(2)}(t) + \dots] < 0, \qquad j = 1,2,\dots,n. \qquad (25)$$

Restricting ourselves in the expansion of the function $\lambda(t)$ to the first two terms of the series, we obtain, according to (23), the simplest approximate stability criteria

$$\mathrm{Re}\{\lambda_j(t) + k_j(t)R[\lambda_j(t),t] - (\xi_j'(t),\eta_j(t))\} < 0, \qquad j = 1,2,\dots,n, \qquad (26)$$

where

$$k_j(t) = (a(t),\eta_j(t))(b(t),\xi_j(t)), \qquad j = 1,2,\dots,n. \qquad (27)$$

Retaining the first three terms of the series in the expansion of the function $\lambda(t)$, we obtain refined stability criteria which take into account, along with the variability of the parameters of the controlled plant, also the variability of the parameters of the automatic-control system. The method presented by us makes it possible in specific cases to analyze the applicability of the widely used method of "frozen" parameters and, if necessary, to introduce the necessary refinements in the investigation of stability.

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