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Abstract

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NUMERICAL WEATHER FORECASTING ON A SPHERE

In solving the problem of forecasting the fields of meteorological elements several days ahead, it becomes necessary in the calculations to use a system of equations for the dynamics of atmospheric processes in a spherical coordinate system. There are various approaches to the solution of the formulated problem ([1], [2]). In the present paper an attempt is made to reduce the problem of weather forecasting on a sphere to a form convenient for computations, and a numerical method is proposed for solving the resulting system of equations by means of a certain set of elementary algorithms ([3], [4]).

The problem of weather forecasting on a spherical Earth in a three-dimensional baroclinic atmosphere, taking into account simplifications natural for atmospheric processes, has the form

$$\frac{dv_\lambda}{dt} + \frac{\operatorname{ctg}\theta}{a} v_\lambda v_\theta + 2\omega\cos\theta\, v_\theta = -\frac{1}{a\sin\theta}\frac{\partial H}{\partial\lambda},$$

$$\frac{dv_\theta}{dt} - \frac{\operatorname{ctg}\theta}{a} v_\lambda^2 - 2\omega\cos\theta\, v_\lambda = -\frac{1}{a}\frac{\partial H}{\partial\theta},$$

$$\frac{dT}{dt} - \frac{\gamma_a}{g} RT\,\frac{\tau}{p} = \frac{\varepsilon}{c_p}, \tag{1}$$

$$\frac{1}{a\sin\theta}\left(\frac{\partial \sin\theta\, v_\theta}{\partial\theta} + \frac{\partial v_\lambda}{\partial\lambda}\right) + \frac{\partial\tau}{\partial p} = 0,$$

$$T = -\frac{p}{R}\frac{\partial H}{\partial p},$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{v_\lambda}{a\sin\theta}\frac{\partial}{\partial\lambda} + \frac{v_\theta}{a}\frac{\partial}{\partial\theta} + \tau\frac{\partial}{\partial p}. \tag{2}$$

Here $v_\lambda, v_\theta, \tau$ are the components of the velocity vector in the coordinate system (λ, θ, p), T is temperature, a is the radius of the Earth, and ε is the heat influx due to nonadiabatic factors. Definitions of the function ε and of other physical quantities are given in ([3]).

To the system of equations (1) it is necessary to append boundary conditions at the surface of the Earth and at the upper boundary of the atmosphere ([3]). As initial data we choose the functions $v_\lambda(\lambda, \theta, p, 0)$, $v_\theta(\lambda, \theta, p, 0)$, $T(\lambda, \theta, p, 0)$.

Let us pass to the local coordinate system (x, y, p) by means of the transformations

$$a \sin\theta \, d\lambda = dx; \qquad a \, d\theta = dy. \tag{3}$$

Analysis of formulas (3) shows that the coordinate y is the direction along the meridian from the pole to the equator, while x is along a circle of latitude. It is important to note that dx is not an independent increment, but depends essentially on $\sin\theta$ (or y). This means that for each circle of latitude the scale of distances will be its own. However, this circumstance is not essential

for the method of solution. In the coordinate system (x, y, p), the system of equations (1) is substantially simplified and takes the form

$$\begin{gathered}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \tau\frac{\partial u}{\partial p} + \chi uv + lv = -\frac{\partial H}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \tau\frac{\partial v}{\partial p} - \chi u^2 - lu = -\frac{\partial H}{\partial y}, \\
\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{\gamma_a - \gamma}{g} RT\,\frac{\tau}{p} = \frac{\varepsilon}{c_p}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \tau}{\partial p} + \chi v = 0, \\
T = -\frac{p}{R}\frac{\partial H}{\partial p},
\end{gathered} \tag{4}$$

where $\chi = \frac{1}{a}\operatorname{ctg}\theta$.

The system of equations (4), with the exception of the terms with coefficient σ, coincides in form with the corresponding system of equations for a plane Earth (3). It should be noted, however, that the similarity is of a formal nature, since in system (4) the domain of definition of the solution is a spherical layer, and the coordinate x is a function of the latitude of the place. It is easy to see that if the sphere is covered by a dense grid of meridians and parallels, then it will be the principal difference grid, supplemented by a system of surfaces through the thickness of the atmosphere.

We shall find the solution of the system of equations (4) by means of a specially defined splitting method, analogous to the case of plane geometry (3). The indicated method makes it possible to approximate the system of equations (4) on the time interval Δt by the following system of equations. On the first partial time interval

$$\begin{gathered}\frac{u^{j+1/5}-u^j}{\Delta t}+\chi\delta_k u^j\left[\beta v^{j+1/5}+(1-\beta)v^j\right]=0,\\ \frac{v^{j+1/5}-v^j}{\Delta t}-\chi\delta_k u^j\left[\beta u^{j+1/5}-(1-\beta)u^j\right]=0,\\ T^{j+1/5}=T^j.\end{gathered} \tag{5}$$

If, further, by φ we denote any function from the set (u, v, T), then on the next three partial intervals we arrive at equations of the same type, of the form

$$\begin{gathered}\frac{\varphi^{j+2/5}-\varphi^{j+1/5}}{\Delta t}+\sigma_k u^j\left[\beta\frac{\varphi_{k+1}^{j+2/5}-\varphi_{k-1}^{j+2/5}}{2\Delta x}+(1-\beta)\frac{\varphi_{k+1}^{j+1/5}-\varphi_{k-1}^{j+1/5}}{2\Delta x}\right]=0,\\ \frac{\varphi^{j+3/5}-\varphi^{j+2/5}}{\Delta t}+\sigma_l v^j\left[\beta\frac{\varphi_{l+1}^{j+3/5}-\varphi_{l-1}^{j+3/5}}{2\Delta y}+(1-\beta)\frac{\varphi_{l+1}^{j+2/5}-\varphi_{l-1}^{j+2/5}}{2\Delta y}\right]=0,\\ \frac{\varphi^{j+4/5}-\varphi^{j+3/5}}{\Delta t}+\sigma_m \tau^j\left[\beta\frac{\varphi_{m+1}^{j+4/5}-\varphi_{m-1}^{j+4/5}}{2\Delta p}+(1-\beta)\frac{\varphi_{m+1}^{j+3/5}-\varphi_{m-1}^{j+3/5}}{2\Delta p}\right]=0.\end{gathered} \tag{6}$$

Consequently, a uniform notation for the equations in the form of system (6) is possible for all functions, with the exception of T. It should be taken into account that in the third equation of system (6), for the function $T^{j+4/5}$, it is necessary to set $\sigma_m\tau^j = 0$, which leads to replacing the indicated equation by the equality

$$T^{j+4/5}=T^{j+3/5}.$$

On the last interval we require the fulfillment of the conditions of divergence of motion and quasistaticity, violated in the process of computation by the formulas

(5), (6). As a result we arrive at the following system of equations, which we shall write in differential-difference form:

$$\begin{gathered}
\frac{u^{j+1}-u^{j+4/5}}{\Delta t}+lv^{j+1}=-H_x^{j+1},\\
\frac{v^{j+1}-v^{j+4/5}}{\Delta t}-lu^{j+1}=-H_y^{j+1},\\
\frac{T^{j+1}-T^{j+4/5}}{\Delta t}-\frac{\gamma_a-\gamma}{g}RT\frac{\tau^{j+1}}{p}=\frac{\varepsilon}{c_p},\\
u_x^{j+1}+u_y^{j+1}+\tau_p^{j+1}+\varkappa v^{j+1}=0,\\
T^{j+1}=-\frac{p}{R}H_p^{j+1}.
\end{gathered}\tag{7}$$

The system of equations (7) is represented in finite-difference form using central differences with respect to all geometric variables. Here β is an arbitrary parameter from the interval $1/2\leqslant\beta\leqslant 1$; the indices k,l,m correspond to the variables x,y,p, respectively; σ is the averaging operator with respect to the index ν,

$$\sigma_\nu\varphi=\frac{1}{2}(\varphi_{\nu+1}+\varphi_{\nu-1}).$$

An essential feature of the split system of equations is that for $\beta=1/2$ it approximates the original system of differential equations (4) up to quantities of the second order of smallness in all variables (5) and does not introduce into the computational scheme the fictitious viscosity that is implicitly present in all difference schemes when $\beta\neq 1/2$. This is especially important in problems of weather forecasting 3–5 days ahead, where loss of accuracy is possible because of the large number of steps in Δt. It can be shown that for β in the interval $1/2\leqslant\beta\leqslant 1$ numerical computation by the difference schemes will be stable for any ratio of the steps.

Special attention must be given to the method for solving the system of equations (7). For this purpose we first eliminate the unknowns $u^{j+1},v^{j+1},\tau^{j+1},T^{j+1}$. Then we arrive at an equation for the function H^{j+1}

$$\frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial H^{j+1}}{\partial p}+\frac{\alpha^2}{1+\alpha^2}\left[H_{xx}^{j+1}+H_{yy}^{j+1}+\varkappa H_y^{j+1}+(\alpha_y+\varkappa\alpha)H_x\right]=-f^{j+1},\tag{8}$$

where

$$\begin{aligned}
f^{j+1}=\frac{\partial}{\partial p}\frac{pR}{m^2}\left(T^{j+4/5}+\alpha\frac{\varepsilon}{c_p l}\right)-\varkappa\frac{\alpha^2}{1+\alpha^2}(v^{j+4/5}+\alpha u^{j+4/5})\\
-\frac{\alpha^2}{1+\alpha^2}\left[(u^{j+4/5}-\alpha v^{j+4/5})_x+(v^{j+4/5}+\alpha u^{j+4/5})_y\right].
\end{aligned}$$

Equation (8) is solved by relaxation methods. In [4] an algorithm was proposed, based on the method of splitting the three-dimensional elliptic operator, which makes it possible to obtain a solution of the problem with accuracy up to quantities of the first order of smallness in Δt. Since in the problem under consideration it is necessary to obtain a solution with accuracy up to quantities of the second order of smallness, it is necessary to use a more accurate method for solving equation (8). As such a method we choose the following.

Let us introduce a new variable ξ, associated with the relaxation parameter $1/\Delta\xi$. Then equation (8) is represented in the form

$$\frac{\partial h}{\partial \xi} = \frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial h}{\partial p} + \frac{\alpha^2}{1+\alpha^2}\left[h_{xx} + h_{yy} + \varkappa h_y + (\alpha_y + \varkappa\alpha)h_x\right] = 0 \tag{9}$$

under the condition

$$h(x, y, p, 0) = f. \tag{10}$$

We integrate equation (9) with respect to ξ over the interval $(0, \infty)$. Then, taking condition (10) into account, we arrive at equation (8) for the function

$$H^{j+1} = \int_0^\infty h\, d\xi. \tag{11}$$

We shall seek the solution of problem (9), (10) by means of the operator-splitting method

$$\begin{aligned}
\frac{h_{n+1/3} - h_n}{\Delta\xi} &= \beta L_1 h_{n+1/3} + (1-\beta) L_1 h_n,\\
\frac{h_{n+2/3} - h_{n+1/3}}{\Delta\xi} &= \beta L_2 h_{n+2/3} - (1-\beta) L_2 h_{n+1/3},\\
\frac{h_{n+1} - h_{n+2/3}}{\Delta\xi} &= \beta L_3 h_{n+1} - (1-\beta) L_3 h_{n+2/3},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
L_1 &= \frac{\alpha^2}{1+\alpha^2}\left[\frac{\partial^2}{\partial x^2} + (a_y + \varkappa a)\frac{\partial}{\partial x}\right],\\
L_2 &= \frac{\alpha^2}{1+\alpha^2}\left(\frac{\partial^2}{\partial y^2} + \varkappa\frac{\partial}{\partial y}\right),\\
L_3 &= \frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial}{\partial p},
\end{aligned}$$

and β is chosen from the interval $1/2 \leq \beta \leq 1$.

After the solution of equations (12) has been obtained under the corresponding boundary conditions (4), the function H^{j+1} is found by means of the finite-difference analogue of relation (11)

$$H^{j+1} = \Delta\xi \left(\sum_{n=0}^{\infty} h_n - \beta f \right). \tag{13}$$

With the aid of harmonic analysis it can be shown that the approximate solution thus obtained, for $\beta \neq 1/2$, approximates the solution of equation (8) with accuracy up to quantities of first order of smallness $\Delta\xi$, and for $\beta = 1/2$, up to quantities of second order. Equations (12) are solved by the finite-difference method, using the factorization method.

After the function H^{j+1} has been found, the remaining unknowns of system (7) are found by elementary operations. Allowance for turbulent exchange and a detailed treatment of the heat influx ε introduce no fundamental changes into the computational scheme.

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REFERENCES

1. E. N. Blinova, DAN, **140**, No. 2 (1961).
2. G. P. Kurbatkin, Izv. AN SSSR, ser. geophys., No. 12 (1962).
3. G. I. Marchuk, DAN, **155**, No. 5 (1964).
4. G. I. Marchuk, DAN, **156**, No. 2 (1964).
5. N. N. Yanenko, DAN, **134**, No. 5 (1960).

Note: Figure translations are in progress. See original paper for figures.