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Abstract

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MATHEMATICS

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CHARACTERIZATION OF REGULAR STRUCTURES BY MEANS OF EXPONENTIAL TOPOLOGY

(Presented by Academician P. S. Aleksandrov on 8 I 1964)

1. Let a distributive structure $\Gamma = (L, \cup, \cap, 0, 1)$ be given. We shall call it a **Urysohn structure** (1) if

$$(a \not\subset b) \to \bigvee_{c} (c \cap a \neq 0)(c \cap b = 0) \tag{1}$$

($\bigvee_c$ denotes "there exists a c such that⋯").

We shall call the structure Γ **regular** (2) if

$$(a \not\subset b) \Rightarrow \bigvee_{cd} (c \cup d = 1)(a \not\subset c)(b \cap d = 0). \tag{2}$$

Denoting by $I(a)$ and $J(a)$ the ideals

$$I(a) = \{x : x \subset a\}, \qquad J(a) = \{x : x \cap a = 0\},$$

we shall call the **exponential topology** for L the topology whose open subbase consists of the sets $L \setminus I(a)$ and $J(a)$, where $a \in L$. In other words, an open base of the space L consists of the sets

$$B(a_0, a_1, \dots, a_n) = J(a_0) \setminus I(a_1) \setminus \dots \setminus I(a_n) =$$
$$= \{x : (x \cap a_0 = 0)(x \not\subset a_1) \cdots (x \not\subset a_n)\}. \tag{3}$$

2. **Theorem.** *Let Γ be a Urysohn structure. In order that the set $F = \{(x, y) : x \subset y\}$ be closed in the space $L \times L$, it is necessary and sufficient that Γ be a regular structure.*

Proof. 1°. The condition is **sufficient**. First of all, it is easy to see that in formula (2) the arrow $\Rightarrow$ may be replaced by the equivalence sign $\equiv$. This makes it possible to obtain the formula

$$L^2 \setminus F = \bigcup_{c \cup d = 1} [L \setminus I(c)] \times J(d).$$

Consequently, the set $L \setminus F$ is open.

2°. The condition is **necessary**.** Let the set F be closed and let $a \not\subset b$. It is required to determine c and d in such a way that

$$c \cup d = 1, \qquad a \not\subset c, \qquad b \cap d = 0. \tag{4}$$

Since the point (a, b) belongs to the open set $L^2 \setminus F$, there exist two sets U and V, belonging to the open base of the space L, such that

$$a \in U, \qquad b \in V, \qquad U \times V \subset L^2 \setminus F.$$

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* This topology coincides with the topology of the structure 2^X of all closed sets of the topological space X (see ([3–5,7])). Ponomarev denotes it by ψX ([6]). It is easy to verify that a topological T_1-space X is regular if and only if the structure 2^X is regular in the sense of formula (2).

** The problem of the necessity of this condition was posed by me in a talk at Moscow State University on 6 XII 1963.

In other words (see (3)): there exist two finite systems $a_0, a_1, \dots, a_m$ and $b_0, b_1, \dots, b_n$ such that:

$$a \cap a_0 = 0, \quad a \not\subset a_i \ (i = 1, \dots, m), \quad b \cap b_0 = 0, \quad b \not\subset b_j \ (j = 1, \dots, n), \tag{5}$$

$$[(x \cap a_0 = 0)(x \not\subset a_i)(y \cap b_0 = 0)(y \not\subset b_j)] \Rightarrow ((x, y) \in L^2 \setminus F) \equiv (x \not\subset y). \tag{6}$$

Let $d = b_0$. We shall prove that among the numbers $1, \dots, m$ there exists an i for which $a_0 \cup a_i \cup b_0 = 1$; thus, taking $c = a_0 \cup a_i$, condition (4) will be proved (on the basis of the first three parts of formula (5)).

Suppose that this is not so; in other words, $a_0 \cup a_i \cup b_0 \neq 1$ for $i = 1, \dots, m$. Substitute in formula (1) $a = 1$ and $b = a_0 \cup a_i \cup b_0$; consequently, there exists c_i such that

$$c_i \neq 0, \quad c_i \cap (a_0 \cup a_i \cup b_0) = 0. \tag{7}$$

Let $x = c_1 \cup \cdots \cup c_m$ and $y = x \cup b$. It is easy to verify (on the basis of (7) and the second half of (5)) that x and y satisfy all the conditions of formula (6) (enclosed in the brackets [ ]). But then, according to formula (6), we get $x \not\subset y$, which clearly contradicts the definition of y.

3. A structure Γ is called a **Brouwer algebra** if in it there exists a difference $x - y$ satisfying the following condition*:

$$(x - y \subset z) \equiv [x \subset (y \cup z)].$$

It is easy to see that if Γ is a regular Brouwer algebra, then the sets $\{(x, y) : x - y = 0\}$ and $\{(x, y) : x - y \subset a\}$ are closed (in $L \times L$). In other words (see (?)), the mapping $x - y$ (from $L \times L$ into L) is lower semicontinuous.

From the theorem proved by us we obtain

Corollary. Let Γ be a Wallman structure and a Brouwer algebra.

Then the following propositions are equivalent:

1°. Γ is a regular structure.

2°. The set $\{(x, y) : x - y = 0\}$ is closed.

3°. The operation $x - y$ is lower semicontinuous.

Recall that for Wallman structures Γ it has been proved (see ([2])) that the following propositions are equivalent:

α) Γ is a normal structure;

β) the set $\{(x, y) : x \cap y = 0\}$ is open;

γ) the operation $x \cap y$ is upper semicontinuous,

where a normal structure is a structure subject to the condition:

$$(a \cap b = 0) \Rightarrow \bigvee_{cd} (c \cup d = 1)(a \cap c = 0 = b \cap d),$$

and upper semicontinuity of the mapping $x \cap y$ means that the set $\{(x, y) : x \cap y \cap a = 0\}$ is open for every $a \in L$.

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* In the case of the structure 2^X, the difference denotes $\overline{A - B}$.

** In the special case of the structure 2^X, where X is a topological T_1-space, this proposition was proved by Engelking ([8]).

Note: Figure translations are in progress. See original paper for figures.