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Abstract

Full Text

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ON COMMUTATIVE ALGEBRAS OF OPERATORS IN THE SPACE Π_1

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Let us consider a commutative algebra R of bounded linear operators in the space Π_k* with indefinite inner product (ξ, η), $\xi, \eta \in \Pi_k$. We shall call the algebra R symmetric if from $A \in R$ it follows that $A^* \in R$, where A^* is defined by the condition $(A\xi, \eta) = (\xi, A^*\eta)$. Two algebras R, R' in two spaces Π_k are called equivalent if there exists a linear mapping of one space onto the other, preserving the indefinite inner product, under which R is mapped onto R'. In the case of ordinary Hilbert spaces, the description, up to equivalence, of all commutative symmetric algebras is well known. It is natural to pose the analogous problem for algebras in Π_k**. The present paper is devoted to the solution of this problem for the simplest case $k = 1$.

Let R be a commutative symmetric algebra in Π_1. According to Theorem 1 of ([4]) (see also ([5])), in Π_1 there exists a one-dimensional nonnegative subspace $\mathfrak{N}$ invariant with respect to all $A \in R$. Let $\xi_0 \in \mathfrak{N}$, $\xi_0 \neq 0$. Then

$$A\xi_0 = \lambda(A)\xi_0 \quad \text{for all } A \in R, \tag{1}$$

where $\lambda(A)$ is a numerical function on R possessing the following properties:

$$\lambda(\alpha A) = \alpha\lambda(A), \qquad \lambda(A + B) = \lambda(A) + \lambda(B),$$

$$\lambda(AB) = \lambda(A)\lambda(B), \qquad |\lambda(A)| \leq |A|. \tag{2}$$

The further classification of the algebras R depends on the properties of the space $\mathfrak{N}$. Only the following cases are possible:

I. Among the nonnegative invariant subspaces $\mathfrak{N}$ there exist positive subspaces (i.e., such that $(\xi, \xi) > 0$ for $\xi \neq 0$, $\xi \in \mathfrak{N}$).

In case I $\lambda(A)$ does not depend on the choice of the positive invariant subspace $\mathfrak{N}$, and for each such subspace $\lambda(A^*) = \overline{\lambda(A)}$. Put

$$\mathfrak{M} = \{\xi : \xi \in \Pi_1,\ A\xi = \lambda(A)\xi\}, \quad \mathfrak{H} = \mathfrak{M}^{\perp}.$$

Then $\mathfrak{M}$ is positive one-dimensional or the space Π_1; $\mathfrak{H}$ is negative (i.e., $(\xi, \xi) < 0$ for $\xi \neq 0$, $\xi \in \mathfrak{H}$) and invariant with respect to all $A \in R$. Let A_1 be the restriction of A to $\mathfrak{H}$, and $R_1 = \{A_1 : A \in R\}$. Then R_1 is a commutative symmetric algebra in an ordinary Hilbert space.

Only the following cases are possible:

Ia. There exist operators $A \in R$ for which $A_1 = 0$, but $\lambda(A) \neq 0$.

Ib. From $A_1 = 0$ it follows that $\lambda(A) = 0$.

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* The study of the space Π_k and of operators in it was begun by L. S. Pontryagin ([1]) and then continued in works of I. S. Iokhvidov and N. G. Krein (in this connection, as well as for definitions and basic properties of Π_k and operators in Π_k, see ([2])).

** For the case of an algebra generated by a single Hermitian (in the sense of (ξ, η)) operator, in contrast to the ordinary Hilbert space this is not always the case in Π_1; this question is closely connected with the question of the spectral representation of Hermitian operators in Π_k, considered by M. G. Krein and G. Langer in ([3]), see also ([2]), Ch. V.

In case Ib the function $\lambda(A)$ may be regarded as a function $\lambda(A_1)$ on R_1, and for $\lambda(A_1)$ the first three conditions (2) are satisfied.

Theorem 1. In case Ia the algebra R is defined by a space $\mathfrak{M}$, one-dimensional positive or of type Π_1, and by a commutative symmetric algebra R_1 in some Hilbert space $\mathfrak{H}$, and in case Ib, in addition, by a numerical function $\lambda(A_1)$ on R_1 satisfying the conditions:

$$\lambda(\alpha A_1) = \alpha\lambda(A_1), \qquad \lambda(A_1+B_1) = \lambda(A_1)+\lambda(B_1), \qquad \lambda(A_1 B_1) = \lambda(A_1)\lambda(B_1),$$
$$\lambda(A_1^*) = \overline{\lambda(A_1)}.$$

Here R is realized in the following way: Π_1 consists of all formal sums $\xi = m+h$, $m \in \mathfrak{M}$, $h \in \mathfrak{H}$, with componentwise definition of addition and multiplication by a number and with scalar product

$$(\xi, \xi') = (m, m') - [h, h']$$

for $\xi = m + h$, $\xi' = m' + h'$, where $[h, h']$ is the ordinary scalar product in $\mathfrak{H}$. The algebra R consists of all operators A defined by the formula

$$A(m + h) = \lambda m + A_1 h, \qquad A_1 \in R_1,$$

where λ is an arbitrary number independent of A_1 in case Ia and $\lambda = \lambda(A_1)$ in case Ib. Two algebras R and $\widetilde{R}$ of type I, corresponding to $\mathfrak{M}, \mathfrak{H}, R_1$ and $\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{H}}, \widetilde{R}_1$ (and to $\lambda(A_1), \tilde{\lambda}(\widetilde{A}_1)$), in case Ib are isomorphic if and only if: a) $\dim \widetilde{\mathfrak{M}} = \dim \mathfrak{M}$; b) $\widetilde{R}_1$ and R_1 are equivalent; c) in case Ib there exists an isometric mapping of $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$, mapping $\widetilde{R}_1$ onto R_1, under which $\tilde{\lambda}(\widetilde{A}_1)$ passes into $\lambda(A_1)$.

II. All nonnegative invariant subspaces $\mathfrak{H}$ are null (i.e. $(\xi, \xi) = 0$ for all $\xi \in \mathfrak{M}$) **and among them there exists an invariant subspace $\mathfrak{N}_1$ such that** $\operatorname{Im} \lambda(A) \neq 0$ **for some Hermitian $A \in R$.**

On the basis of Theorem 2 from ([6]), in this case there exists a null invariant subspace $\mathfrak{N}_2$, skew-conjugate to $\mathfrak{N}_1$, and such that

$$A\xi = \overline{\lambda(A)}\, \xi$$

for $\xi \in \mathfrak{N}_2$, $A = A^* \in R$. It is not hard to show that the pair $\mathfrak{N}_1, \mathfrak{N}_2$ is determined uniquely up to interchange. Obviously, one can choose $\xi_1 \in \mathfrak{N}_1$ and $\xi_2 \in \mathfrak{N}_2$ so that $(\xi_1, \xi_2) = 1$. Further, $\mathfrak{N}_1 \dot{+} \mathfrak{N}_2$ is an invariant subspace of type Π_1. Therefore, putting

$$\mathfrak{H} = (\mathfrak{N}_1 \dot{+} \mathfrak{N}_2)^{\perp},$$

we obtain that

$$\Pi_1 = (\mathfrak{N}_1 \dot{+} \mathfrak{N}_2) \oplus \mathfrak{H},$$

$\mathfrak{H}$ is negative and invariant with respect to all $A \in R$. Let A_1 be the restriction of A to $\mathfrak{H}$ and

$$R_1 = \{A_1 : \ A \in R\}.$$

Then R_1 is a symmetric commutative algebra in an ordinary Hilbert space.

Only the following two cases are possible:

IIa. There exist operators $A \in R$ for which $A_1 = 0$, but $\lambda(A) \neq 0$.

IIb. From $A_1 = 0$ it follows that $\lambda(A) = 0$.

In case IIb the function $\lambda(A)$ may be regarded as a function $\lambda(A_1)$ on R_1.

Theorem 2. In case IIa the algebra R is defined by a commutative symmetric algebra R_1 of operators in some Hilbert space $\mathfrak{H}$, and in case IIb, in addition, by a function $\lambda(A_1)$, defined on R_1 and satisfying the conditions:

$$\lambda(\alpha A_1) = \alpha\lambda(A_1), \qquad \lambda(A_1 + B_1) = \lambda(A_1) + \lambda(B_1),$$

$$\lambda(A_1 B_1) = \lambda(A_1)\lambda(B_1), \qquad \lambda(A_1^*) \neq \overline{\lambda(A_1)}.$$

The algebra R is realized in the following way: Π_1 is the space of all formal sums

$$\xi = \alpha_1\xi_1 + \alpha_2\xi_2 + h,$$

where $h \in \mathfrak{H}$; α_1, α_2 are arbitrary complex numbers; ξ_1, ξ_2 are abstract elements. The operations of addition and multiplication by a number are defined in Π_1 componentwise, and the scalar product is given by the formula

$$(\xi, \xi') = \alpha_1\overline{\alpha_2'} + \alpha_2\overline{\alpha_1'} - [h, h']$$

for

$$\xi = \alpha_1\xi_1 + \alpha_2\xi_2 + h, \qquad \xi' = \alpha_1'\xi_1 + \alpha_2'\xi_2 + h',$$

where $[h, h']$ is the ordinary scalar product in $\mathfrak{H}$.

The algebra R consists of all linear operators A in Π_1 defined by the formulas

$$A\xi_1 = \lambda\xi_1, \qquad A\xi_2 = \mu\xi_2, \qquad Ah = A_1 h,$$

where $A_1 \in R_1$; λ, μ are arbitrary complex numbers in case IIa and $\lambda = \lambda(A_1)$, $\mu = \overline{\lambda(A_1^*)}$ in case IIb. Two algebras R and $\widetilde{R}$ of type II, corresponding to R_1 and $\widetilde{R}_1$ (and to $\lambda(A_1), \tilde{\lambda}(\widetilde{A}_1)$ in case IIb), are equivalent if and only if R_1 and $\widetilde{R}_1$ are equivalent, and in the case-

where IIb means when there exists an isometric mapping $\mathfrak{H}$ onto $\mathfrak{H}$, under which $\widetilde{R}_1$ is mapped onto R_1 and $\tilde{\lambda}(\widetilde{A}_1)$ passes into $\lambda(A_1)$ or $\overline{\lambda(A^*)}$.

III. **All nonnegative invariant subspaces $\mathfrak{N}$ are null, and for each of them $\lambda(A^*) = \overline{\lambda(A)}$.** It is not difficult to show that in this case $\mathfrak{N}$ is unique and, consequently, $\lambda(A)$ is determined uniquely. In case III we shall additionally assume that $1 \in R$, that Π_1 is separable, and that R is separable in the operator norm. Let $\xi_0 \in \mathfrak{N}$, $\xi_0 \neq 0$, and let η_0 be such that $(\eta_0, \eta_0) = 0$, $(\xi_0, \eta_0) = 1$. Put $\mathfrak{H} = \{\xi_0, \eta_0\}^{\perp}$. Then $\mathfrak{H}$ is negative, and Π_1 consists of all sums $\alpha\xi_0 + \beta\eta_0 + h$, where α, β are arbitrary complex numbers and $h \in \mathfrak{H}$. The subspace $\mathfrak{H}$ is no longer invariant with respect to the operators $A \in R$, but

$$Ah = (h, h_A)\xi_0 + A_1 h,$$

where $h_A \in \mathfrak{H}$ and A_1 is a bounded linear operator in $\mathfrak{H}$. Put $R_1 = \{A_1 : A \in R\}$. Then R_1 is a symmetric commutative algebra in the usual Hilbert space $\mathfrak{H}$, containing the identity operator; it is, up to equivalence, determined uniquely, independently of the choice of η_0.

Let $\widehat{R}_1$ be the closure of R_1 in the operator norm, and let T be the bicompact space of maximal ideals t of the algebra $\widehat{R}_1$. Let $A(t)$ be the value of the element $A_1 \in R$ on the ideal t. Only the following two cases are possible:

IIIa. **There exists a point $t_0 \in T$ for which $A(t_0) = \lambda(A)$ for all $A \in R$.**

IIIb. **Such a point $t_0 \in T$ does not exist.**

Theorem 3. *In case IIIa a separable algebra R with identity in the separable space Π_1 is specified by:*

a) *a bicompact space T;*

b) *a self-adjoint algebra R_1 of numerical functions $A(t) \in C(T)$, everywhere dense in $C(T)$;*

c) *a measure σ with support T;*

d) *a finite or countable system of closed sets $F_1 = T \supset F_2 \supset \cdots$, a point $t_0 \in T$, and the Hilbert space $\mathfrak{P}$ constructed from them, consisting of all* *vector-functions* $p = \{p(t)\} = \{p_1(t), p_2(t), ...\}$, $t \in T_1 = T - \{t_0\}$, *where* $p_k(t) \in L^2_\sigma(T_1)$ *and* $p_k(t) = 0$ *almost everywhere on* $T - F_k$, *with the usual definition of the operations of addition and multiplication by a number and with scalar product*

$$[p, p'] = \int (p(t), p'(t))\, d\sigma = \int \sum_k p_k(t)\, \overline{p'_k(t)}\, d\sigma,$$

where everywhere the integral is taken over T_1.

Denote by $\mathfrak{P}(t)$ the subspace in l^2 consisting of all vectors $p = \{p_1, p_2, ...\} \in l^2$ for which $p_k = p_{k+1} = \cdots = 0$ when $t \in T_1 - F_k$. Then

$$\mathfrak{P} = \int \mathfrak{P}(t)\, d\sigma.$$

If $T_1 = \emptyset$, then, by definition, $\mathfrak{P} = (0)$.

R is specified by:

d) *a Hilbert space* $\mathfrak{L}$ *(possibly* $= (0)$*), in it a linear subset* $\mathcal{L}$ *(possibly* $= (0)$*) and an anti-isometric operator** V, *mapping* $\mathcal{L}$ *onto* $\mathcal{L}$ *and satisfying the condition* $V^2 = 1$;

e) *a measurable vector-function* $\zeta = \{\zeta(t)\}$ *such that* $\zeta(t) \in \mathfrak{P}(t)$ *almost everywhere on* T_1 *and*

$$\{[A(t) - A(t_0)]\zeta(t)\} \in \mathfrak{P}$$

for any function $A(t) \in R_1$;

f) *a linear manifold* $\mathcal{E}$ *in the collection* $\mathfrak{E}$ *of all* $\{A(t), q, \gamma, \lambda\}$, *where* $A(t) \in R_1$, $q \in \mathcal{L}$, $\gamma \in C$ *(*C *is the set of all complex numbers),* $\lambda = A(t_0)$, *satisfying the following conditions: 1) if* $\{A(t), q, \gamma, \lambda\} \in \mathcal{E}$, *then also* $\{\overline{A(t)}, Vq, \overline{\gamma}, \overline{\lambda}\} \in \mathcal{E}$; *2) if* $\{A_1(t), q_1, \gamma_1, \lambda_1\} \in \mathcal{E}$ *and* $\{A_2(t), q_2, \gamma_2, \lambda_2\} \in \mathcal{E}$, *then also*

$$\left\{A_1(t)A_2(t),\ \overline{\lambda_2}q_1 + \overline{\lambda_1}q_2,\ \lambda_1\gamma_2 + \lambda_2\gamma_1 - \int |A_1(t) - \lambda_1|\,|A_2(t) - \lambda_2|\,(\zeta(t), \zeta(t))\, d\sigma + (Vq_2, q_1),\ \lambda_1\lambda_2\right\} \in \mathcal{E}$$

3) the projection of $\mathcal{E}$ *onto the first, second, third, and fourth components coincides respectively with* R_1, $\mathcal{L}$, C, *and* C.

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* An operator V in $\mathcal{L}$ is called **anti-isometric** if

$$V(\alpha_1 q_1 + \alpha_2 q_2) = \overline{\alpha_1} V q_1 + \overline{\alpha_2} V q_2$$

and

$$(Vq_1, Vq_2) = (q_2, q_1)$$

for $q_1, q_2 \in \mathcal{L}$, $\alpha_1, \alpha_2 \in C$.

The algebra R is realized as follows. The space Π_1 consists of all formal sums $\xi = \alpha\xi_0 + \beta\eta_0 + p(t) + q$, where $\alpha, \beta \in C$, $p(t) \in \mathfrak{P}$, $q \in \mathfrak{Q}$, ξ_0, η_0 are abstract elements, with componentwise definitions of the operations of addition and multiplication by a number, and with scalar product

$$(\xi, \xi') = \alpha\bar{\beta}' + \bar{\beta}\alpha' - \int (p(t), p'(t))\, d\sigma - [q, q']$$

for $\xi = \alpha\xi_0 + \beta\eta_0 + p(t) + q$, $\xi' = \alpha'\xi_0 + \beta'\eta_0 + p'(t) + q'$, where $[q, q']$ is the usual scalar product in $\mathfrak{Q}$.

The algebra R consists of all linear operators A in Π_1 given by the formulas

$$A\xi_0 = \lambda\xi_0, \qquad Ap(t) = -\int (A(t) - \lambda)(p(t), \zeta(t))\, d\sigma\, \xi_0 + A(t)p(t); \tag{3}$$

$$Aq = (q, q')\xi_0 + \lambda q; \tag{4}$$

$$A\eta_0 = \gamma\xi_0 + \lambda\eta_0 + (A(t) - \lambda)\zeta(t) + Vq, \tag{5}$$

where $\{A(t), q', \gamma, \lambda\} \in \mathfrak{E}$.

Two such algebras $R, \widetilde{R}$ with the same $\xi_0, \eta_0, T, t_0, \mathfrak{P}$, and $\mathfrak{Q}$, but corresponding, possibly, to different $R_1, \zeta(t), \mathfrak{L}, V, \mathfrak{E}$ and $\widetilde{R}_1, \tilde{\zeta}(t), \widetilde{\mathfrak{L}}, \widetilde{V}, \widetilde{\mathfrak{E}}$, respectively, are equivalent if and only if there exist: 1) a homeomorphism s of the space T onto itself; 2) a measurable family, defined for almost all $t \in T_1$, of isometric mappings $U(t)$ of the space $\mathfrak{P}(st)$ onto $\mathfrak{P}(t)$; 3) a unitary operator U_0 in $\mathfrak{Q}$; 4) elements $p_0(t) \in \mathfrak{P}$ and $q_0 \in \mathfrak{Q}$ such that: α) $st_0 = t_0$; β) the correspondence $A(t) \to A(st)$ is a mapping of the algebra R_1 onto $\widetilde{R}_1$; γ) the measures $\sigma(\Delta)$ and $\sigma(s\Delta)$ are equivalent; δ) U_0 maps $\mathfrak{L}$ onto $\widetilde{\mathfrak{L}}$ and, moreover, V is transformed into $\widetilde{V}$; ε)

$$\tilde{\zeta}(t) = \left[\frac{d\sigma(st)}{d\sigma(t)}\right]^{1/2} U(t)[\zeta(st) + p_0(st)];$$

χ) the correspondence $\{A(t), q, \gamma, \lambda\} \to \{\widetilde{A}(t), \tilde{q}, \tilde{\gamma}, \tilde{\lambda}\}$, where $\widetilde{A}(t) = A(st)$, $\tilde{q} = U_0 q$,

$$\tilde{\gamma} = \gamma - \int (A(t) - \lambda)\left[(\zeta(t), p_0(t)) + (p_0(t), \zeta(t)) + (p_0(t), p_0(t))\right] d\sigma + (Vq', q_0) + (q_0, Vq')$$

is a mapping of the set $\mathfrak{E}$ onto the set $\widetilde{\mathfrak{E}}$.

In case IIIb the algebra R is realized analogously, with the difference that now $t_0, \mathfrak{Q}, V, U_0, q_0$ are absent, formula (4) and the condition $st_0 = t_0$ are absent; T_1 coincides with T, and λ runs through arbitrary complex numbers no longer necessarily equal to $A(t_0)$.

Corollary 1. In case IIIb one may assume that $\zeta(t) \equiv 0$, and when $\zeta(t) \equiv 0$ formulas (3)–(5) for the operators $A \in R$, including the coefficient γ, are determined uniquely.

Corollary 2. In cases I and II the algebras R are semisimple; in case IIIa the radical of the algebra R consists of those and only those operators $A \in R$ for which $A^3 = 0$ (in case IIIb, for which $A^2 = 0$).

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Note: Figure translations are in progress. See original paper for figures.