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## Abstract

## Full Text

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# THE AREA METHOD IN THE THEORY OF UNIVALENT FUNCTIONS

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We consider the following classes of functions: $\Sigma$—the class of functions

$$F(\zeta) = \zeta + \alpha_0 + \alpha_1 \frac{1}{\zeta} + \cdots,$$

regular and univalent in the domain $1 < |\zeta| < \infty$; the classes $\Sigma_0, \Sigma_2, S, S_2$ are defined as usual; $\widetilde{\Sigma}, \widetilde{\Sigma}_0, \widetilde{\Sigma}_2$ are subclasses, respectively, of $\Sigma, \Sigma_0, \Sigma_2$, consisting of functions mapping $|\zeta| > 1$ onto domains without exterior points with boundary of zero area (which is equivalent to the condition $\sum_{n=1}^{\infty} n|\alpha_n|^2 = 1$).

The paper presents a method related to an elementary one, but which makes it possible to obtain, in a unified way, the previously known principal results of the theory of univalent functions and some new results.

## I. The system of functions $\{A_n(z)\}$

### 1. Simplest properties of the system $\{A_n(z)\}$.

Take an arbitrary function $F(\zeta) \in \Sigma$ and an arbitrary value $\zeta_1$: $|\zeta_1| = \rho = 1/r > 1$. Define in the disk $|z| < 1$ single-valued functions $A_n(z)$, $n = 1, 2, \ldots$, by means of the expansion

$$\ln \frac{F(\zeta) - F(\zeta_1)}{\zeta - \zeta_1} = \sum_{n=1}^{\infty} A_n(z_1) \frac{1}{\zeta^n}, \qquad z_1 = \frac{1}{\zeta_1}, \tag{1}$$

(the branch of the logarithmic function is taken which tends to zero as $\zeta = \infty$).

If we take the function $F_2(\zeta) = \sqrt{F(\zeta^2)} \in \Sigma_2$, when $F(\zeta) \in \Sigma_0$, and in the same way define the system of functions $\{a_n(z)\}$:

$$\ln \frac{F_2(\zeta) - F_2(\zeta_1)}{\zeta - \zeta_1} = \sum_{n=1}^{\infty} a_n(z_1) \frac{1}{\zeta^n}, \qquad z_1 = \frac{1}{\zeta_1}, \tag{2}$$

then the systems $\{A_n(z)\}$ and $\{a_n(z)\}$ are related by

$$A_n(z^2) = 2a_{2n}(z), \qquad n = 1, 2, \dots . \tag{3}$$

The simplest properties of the system $\{A_n(z)\}$ are easily established:

a) The functions $A_n(z)$, $n = 1, 2, \dots$, are regular in the disk $|z| < 1$. If the functions $A_n(z)$, $n = 1, 2, \dots$, are expanded in Taylor series in a neighborhood of $z = 0$,

$$A_n(z) = \sum_{k=1}^{\infty} \alpha_{nk} z^k, \qquad n = 1, 2, \dots , \tag{4}$$

then $\alpha_{nk} = \alpha_{kn}$, $n, k = 1, 2, \dots$.

b) The system of functions $\{A_n(z)\}$ is uniformly bounded in the disk $|z| < 1$, i.e., for every $z$ in the disk $|z| < 1$ and $n = 1, 2, \dots$ we have

$$|A_n(z)| < C, \tag{5}$$

where $C$ is an absolute constant.

c) The system $\{A_n(z)\}$, generated by the function $w = F(\zeta) \in \Sigma$, is connected with the system of Faber polynomials $\Phi_n(w)$, generated by the function $\zeta = \Phi(w)$, inverse to the function $w = F(\zeta)$, by the equalities

$$nA_n(z) = \zeta^n - \Phi_n(F(\zeta)), \qquad z = \frac{1}{\zeta}, \qquad n = 1, 2, \dots . \tag{6}$$

## 2. Orthogonality of the system $\{A_n(z)\}$

Deeper properties of the system $\{A_n(z)\}$ are obtained with the aid of the following lemma.

**Lemma.** Let

$$F(\zeta) = \zeta + a_0 + a_1 \frac{1}{\zeta} + \cdots \in \Sigma,$$

and let $Q(w)$ be an arbitrary function, different from a constant, regular in the domain $g_{\rho_0}$ ($g_{\rho_0}$ is the interior of the image of the circle $|\zeta| = \rho_0 > 1$ under the mapping by the function $F(\zeta)$). Suppose that the Laurent expansion of the function $Q(F(\zeta))$, regular in the annulus $1 < |\zeta| < \rho_0$, has the form

$$Q(F(\zeta)) = \sum_{n=1}^{\infty} \beta_n \frac{1}{\zeta^n} + b_0 + \sum_{n=1}^{\infty} b_n \zeta^n, \qquad 1 < |\zeta| < \rho_0. \tag{7}$$

Then

$$\sum_{n=1}^{\infty} n|\beta_n|^2 \leq \sum_{n=1}^{\infty} n|b_n|^2. \tag{8}$$

The equality sign occurs if and only if $F(\zeta) \in \widetilde{\Sigma}$.

This lemma, established earlier by the author ([1]) without the conclusion concerning the possibility of the equality sign in (8), is a generalization of the "external area theorem." Applying it to $F(\zeta) \in \widetilde{\Sigma}$ and choosing $Q(w) = \lambda\Phi_n(w) + \Phi_m(w)$, $n, m = 1, 2, \ldots$, $\lambda$ an arbitrary complex number (or $Q(w) = \lambda \ln(w - w_1) + \ln(w - w_2)$, $w_1 = F(\zeta_1)$, $w_2 = F(\zeta_2)$), we arrive at the theorem:

**Theorem 1.** For any system

$$\left\{ A_n(z) = \sum_{k=1}^{\infty} a_{nk} z^k \right\}, \qquad n = 1, 2, \ldots,$$

generated by a function $F(\zeta) \in \widetilde{\Sigma}$, the system of derivatives $\{A_n'(z)\}$ is orthogonal in the domain $|z| < 1$. Moreover, the equalities hold

$$\frac{1}{\pi} \iint_{|z|<1} A_n'(z)\, \overline{A_m'(z)}\, d\sigma = \sum_{k=1}^{\infty} k a_{nk} \overline{a_{mk}} = \begin{cases} 0, & \text{if } m \neq n, \\ \dfrac{1}{n}, & \text{if } m = n. \end{cases} \tag{9}$$

It follows from Theorem 1 that the system of functions $\{\sqrt{n}\, A_n(z)\}$, generated by a function $F(\zeta) \in \widetilde{\Sigma}$, is orthonormal in the domain $|z| < 1$. It is not difficult to prove the completeness of this system in the class of functions regular in the unit disk and with integrable square of the modulus in $|z| < 1$. This makes it possible to formulate a theorem on the expandability of regular functions in the system $\{A_n(z)\}$.

**Theorem 2.** If the system $\{A_n(z)\}$ is generated by a function $F(\zeta) \in \widetilde{\Sigma}$, then for any function

$$f(z) = \sum_{n=1}^{\infty} c_n z^n$$

regular in the disk $|z| < 1$ and with finite area of the image of this disk, there is an expansion in the system $\{A_n(z)\}$ (10) and the area formula (11):

$$f(z) = \sum_{n=1}^{\infty} \lambda_n A_n(z), \qquad \lambda_n = \frac{n}{\pi} \iint_{|z|<1} f'(z)\, \overline{A_n'(z)}\, d\sigma; \tag{10}$$

$$\sum_{n=1}^{\infty} n|c_n|^2 = \sum_{n=1}^{\infty} \frac{|\lambda_n|^2}{n}. \tag{11}$$

Moreover, the series $\sum_{n=1}^{\infty} \lambda_n A_n(z)$ converges absolutely and uniformly inside the circle $|z| < 1$.

Finally, using the orthonormality of the system $\{\sqrt{n}\, A_n(z)\}$, we obtain the sharp estimates (12) and (13).

**Theorem 3.** For any systems $\{A_n(z)\}$ and $\{a_n(z)\}$, $n = 1, 2, ...,$ generated respectively by the functions $F(\zeta) \in \Sigma$ and $F_2(\zeta) \in \Sigma_2$, the following inequalities hold in the circle $|z| < 1$:

$$\sum_{n=1}^{\infty} n|A_n(z)|^2 \leq \ln \frac{1}{1 - r^2}, \qquad r = |z|; \tag{12}$$

$$\sum_{n=1}^{\infty}(2n-1)|a_{2n-1}(z)|^2 \leq \frac{1}{2} \ln \frac{1 + r^2}{1 - r^2}, \qquad \sum_{n=1}^{\infty} 2n|a_{2n}(z)|^2 \leq \frac{1}{2} \ln \frac{1}{1 - r^4}, \qquad r = |z|. \tag{13}$$

Equality in (12) holds if and only if $F(\zeta) \in \widetilde{\Sigma}$, and in (13) if $F_2(\zeta) \in \widetilde{\Sigma}_2$.

## II. Univalent functions

We indicate some applications of the remarkable properties of the system $\{A_n(z)\}$.

**1. Distortion theorems.** From the definition of the systems $\{A_n(z)\}$ and $\{a_n(z)\}$ we obtain the following expressions for certain functionals in the classes $\Sigma_0$ and $S$:

$$\ln F'(\zeta) = \sum_{n=1}^{\infty} A_n(z) z^n;$$

$$\ln \frac{F_2(\zeta)}{\zeta} = \ln \frac{z}{f_2(z)} = \sum_{n=1}^{\infty} (-1)^n a_n(z) z^n, \qquad z = \frac{1}{\zeta}; \tag{14}$$

$$\ln \frac{\zeta F_2'(\zeta)}{F_2(\zeta)} = \ln \frac{z f_2'(z)}{f_2(z)} = 2 \sum_{n=1}^{\infty} a_{2n-1}(z) z^{2n-1}.$$

Estimating the modulus of each of the sums in (14) by means of Cauchy' s inequality and taking into account the estimates (12) and (13), we obtain the classical distortion theorems. The Goluzin "distortion of chords" theorems ([2]) and Bazilevich's "distortion of central angles"theorems ([3]) are proved analogously. Thus Theorem 3 is a universal distortion theorem in the classes $S$ and $\Sigma$.

From the orthonormality of the system $\{\sqrt{n}\, A_n(z)\}$ there also easily follow the generalized distortion-type theorems obtained by G. M. Goluzin.

**2. Conditions for univalence.** Let the function $F(\zeta)$ be given in the domain $|\zeta| > 1$ by the formal Laurent series

$$F(\zeta) = \zeta + a_0 + \sum_{n=1}^{\infty} a_n \zeta^{-n}. \tag{15}$$

In the exterior of the unit circle form the formal expansion of the function $\ln \dfrac{F(\zeta) - F(t)}{\zeta - t}$ in a double power series:

$$\ln \frac{F(\zeta) - F(t)}{\zeta - t} = \sum_{n,k=1}^{\infty} \alpha_{nk} \zeta^{-n} t^{-k}. \tag{16}$$

where $a_{nk}$ are integral rational functions of $a_1, a_2, \dots, a_m$, $m = \max(n, k)$. Then Theorem 1 is not difficult to reformulate in the form of conditions for the univalence of $F(\zeta)$.

**Theorem 4.** In order that the function $F(\zeta)$, given by formula (15), be regular and univalent in the domain $1 < |\zeta| < \infty$, it is necessary and sufficient that the coefficients $a_{nk}$ of the formal expansion (16) satisfy the system of inequalities:

$$\sum_{k=1}^{\infty} k |a_{nk}|^2 \leqslant \frac{1}{n}, \qquad n = 1, 2, \dots \tag{17}$$

Moreover, if in (17) the equality sign holds for at least one value of $n$, then $F(\zeta) \in \widetilde{\Sigma}$ (and the equality sign in (17) will hold for every $n \geqslant 1$).

For the coefficients $a_{nk}$ $(n, k = 1, 2, \dots)$, generated by functions $F(\zeta) \in \widetilde{\Sigma}$, for arbitrary values of the complex variables $x_n$ $(n = 1, 2, \dots, m)$, $m \geqslant 1$, from Theorem 2 we obtain the identity

$$\sum_{n=1}^{\infty} n \left| \sum_{k=1}^{m} x_k a_{nk} \right|^2 = \sum_{n=1}^{m} \frac{|x_n|^2}{n}. \tag{18}$$

Identity (18) immediately leads to the estimate

$$\left| \sum_{n,k=1}^{m} a_{nk} x_{nx} k \right| \leqslant \sum_{n=1}^{m} \frac{|x_n|^2}{n}, \tag{19}$$

obtained by M. Schiffer[4] by the method of variations and by G. M. Goluzin[5] by Löwner’ s parametric method.

## 3. Mean integral modulus.

**Theorem 5.** For the function $f(z) = \overline{f(\overline{z})} \in S$, for any $r$, $0 < r < 1$, we have the sharp estimate

$$\frac{1}{2\pi}\int_{|z|=r} |f(z)|\, d\arg z \leqslant \frac{r}{1-r^2}. \tag{20}$$

The equality sign occurs only for the functions

$$f(z) = \frac{z}{(1\pm z)^2}.$$

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*Note: Figure translations are in progress. See original paper for figures.*