G. A. ZAITSEV

1964

SovietRxiv

Abstract

Full Text

G. A. ZAITSEV

BASIC FORMULAS FOR A MULTIDIMENSIONAL REAL SPINOR AND AN ALGEBRAIC MODEL OF QUANTIZED WAVE FIELDS

(Presented by Academician N. N. Bogolyubov, January 14, 1964)

The theory of quantized wave fields is based on the application of creation and annihilation operators for fermions and bosons ($^{1-3}$). These operators generate an abstract associative ring whose generators, for the discrete momentum representation, are related by the relations

$$b^{\sigma_1} b_{\sigma_2} + b_{\sigma_2} b^{\sigma_1} \equiv [b^{\sigma_1}, b_{\sigma_2}]_+ = \delta^{\sigma_1}_{\sigma_2} e, \qquad [b_{\sigma_1}, b_{\sigma_2}]_+ = [b^{\sigma_1}, b^{\sigma_2}]_+ = 0; \quad (1)$$

$$c^{\mu_1} c_{\mu_2} - c_{\mu_2} c^{\mu_1} \equiv [c^{\mu_1}, c_{\mu_2}] = \delta^{\mu_1}_{\mu_2} e, \qquad [c_{\mu_1}, c_{\mu_2}] = [c^{\mu_1}, c^{\mu_2}] = 0. \quad (2)$$

Here e is the element of the ring corresponding to the identity operator; b_σ and b^σ correspond to the fermion creation and annihilation operators, c_μ and c^μ to the boson creation and annihilation operators, which commute with b_σ, b^σ; the indices σ and μ run through countable sequences of values. The operators corresponding to elements of the abstract ring will be denoted by the same Latin letters, but no longer lowercase; rather, uppercase.

Definition 1. The associative rings generated by the elements b_σ, b^σ or c_μ, c^μ with the commutation relations (1) or (2) will be called, respectively, the **basic rings of a quantized fermion field** $\{b\}$ and **boson field** $\{c\}$. By the symbol $\{b \times c\}$ we denote the direct product of the rings $\{b\}$ and $\{c\}$, and by the symbols $\{b \times P\}$, $\{c \times P\}$, $\{b \times c \times P\}$ the associative algebras of the indicated rings over the field P (see (4)).

For the solution of mathematical problems that have been discussed repeatedly (^{2,3,5,6}), connected with a deeper foundation of the theory of quantized fields, as well as with an invariant-group restructuring of the theory, it is desirable to find and study a clear algebraic model corresponding to the case when the indices σ and μ take only a finite number of integer values, for example lying between 1 and n_b and n_c. It will be shown below that such a model can be furnished by a correspondingly generalized theory of real spinors (^{7}), and that the basic formulas for one real spinor (⁸) can be given the meaning of equations for determining the vacuum.

Introduce $2n_b$ linearly independent real matrices R_α, related by the relations

$$R_{\alpha_1}R_{\alpha_2}+R_{\alpha_2}R_{\alpha_1}=2\varepsilon_{\alpha_1}\delta_{\alpha_1\alpha_2}E, \qquad \varepsilon_1=\varepsilon_2=\dots=\varepsilon_{n_b+1}=1, \qquad \varepsilon_{n_b+2}=\dots=\varepsilon_{2n_b}=-1. \tag{3}$$

Choose such a special representation that the matrices $S_0=R_1$, $S_\sigma=R_{1+\sigma}$ are symmetric, while $A_1=R_{n_b+2},\dots,A_{n_b-1}=R_{2n_b}$, $A_{n_b}=(-1)^{n_b-1}R_1\cdots R_{2n_b}$ are antisymmetric, and the elements of these matrices are equal to $0, 1$, or -1. The existence of such a representation is easily proved by induction on n_b. Instead of the matrices R_α, as generators of the ring $\{R\}$, take $2n_b$ linearly independent matrices S_σ, A_σ of order 2^{n_b}. It follows from (3) that, if one puts

$$S_\sigma^+=\begin{pmatrix}S_\sigma & 0\\ 0 & S_\sigma\end{pmatrix}, \qquad A_\sigma^+=\begin{pmatrix}A_\sigma & 0\\ 0 & A_\sigma\end{pmatrix}, \qquad S_{n_b+1}^+=\begin{pmatrix}0 & S_0\\ S_0 & 0\end{pmatrix}, \qquad A_{n_b+1}^+=\begin{pmatrix}0 & -S_0\\ S_0 & 0\end{pmatrix}, \tag{4}$$

then the matrices with the "plus" sign will obey relations of type (3) with n_b replaced by n_b+1. Taking, for $n_b=1$,

$$S_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad A_1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

with the aid of (4) we shall find, for arbitrary n_b, the simplest representation of the ring $\{R\}$ with the indicated properties, which we shall distinguish from other representations by a zero on the left.

Definition 2. The associative ring $\{R\}$ generated by the real matrices R_a related to one another by the relations (3) will be called the **basic ring of real spinors**. The associative algebra of this ring over the field P will be denoted by $\{R\times P\}$.

Introducing linearly independent matrices B^σ and B_σ by the formulas

$$B^\sigma=\frac{1}{2}(S_\sigma-A_\sigma), \qquad B_\sigma=\frac{1}{2}(S_\sigma+A_\sigma), \tag{5}$$

with the aid of (3) we obtain

$$B^{\sigma_1}B_{\sigma_2}+B_{\sigma_2}B^{\sigma_1}=[B^{\sigma_1},B_{\sigma_2}]_+=\delta^{\sigma_1}_{\sigma_2}E, \qquad [B_{\sigma_1},B_{\sigma_2}]_+=[B^{\sigma_1},B^{\sigma_2}]_+=0. \tag{6}$$

Comparison with (1) shows that the following is true.

Theorem 1. *The basic algebra of real spinors over an arbitrary field P of characteristic zero is isomorphic to the basic algebra of a fermion field over P.*[1]

Generalizing to the multidimensional case the basic formulas for a single real spinor ([8]), we introduce a column ψ_0 of 2^{n_b} real numbers such that

$$\frac{1}{2}(S_\sigma - A_\sigma)\psi_0 \equiv B^\sigma \psi_0 = 0, \qquad \sigma = 1, 2, \dots, n_b, \qquad \psi_0 \neq 0. \tag{7}$$

Definition 3. The real column ψ_0, determined as a result of solving the equations (7), will be called a **basic real spinor**.

The law of transformation of ψ_0 with respect to one or another group of transformations is determined by the law of transformation of the basic matrices B^σ that preserves the relations (6). For $n_b = 2$, hence, in particular, upon postulating the corresponding transformation properties of the matrices B^σ, one obtains the known law of transformation of the real spinor ψ_0 with respect to transformations from the Lorentz group ([7; 8]). If one introduces the additional normalization condition $\psi_0'\psi_0 = 1$, then (7) will be a finite-dimensional analogue of the equations for determining the vacuum.

Theorem 2. *If P is a field of characteristic zero and there are two systems ${}_0B_\sigma, {}_0B^\sigma$ and B_σ, B^σ of matrices of order 2^{n_b} with elements from the field P, related by the relations (6), then $B_\sigma = S({}_0B_\sigma)S^{-1}$, $B^\sigma = S({}_0B^\sigma)S^{-1}$, where S is a nonsingular square matrix with elements from P. The algebra $\{b \times P\}$ is isomorphic to the algebra of square matrices of order 2^{n_b} over the field P.*

Theorem 3. *By specifying a representation of the ring of a fermion field, the basic real spinor is determined up to a factor.*

Taking account of Theorem 2 and formulas (7), a general 2^{n_b}-dimensional column ψ can be expressed in the following way through the column ψ_0 and the matrices B:

$$\psi = \left(\psi^{[0]}E + \frac{1}{1!}\psi^{[\sigma_1]}B_{\sigma_1} + \frac{1}{2!}\psi^{[\sigma_1\sigma_2]}B_{\sigma_1}B_{\sigma_2} + \dots + \frac{1}{n_b!}\psi^{[\sigma_1\cdots\sigma_{n_b}]}B_{\sigma_1}\cdots B_{\sigma_{n_b}}\right)\psi_0,$$

where, as usual, summation over identical indices is understood.

(8)

Definition 4. A column ψ, expressed in terms of the basic real spinor by formula (8), will be called a **real spinor** if the law of transformation of ψ under

---

[1] After the present article had been written, the author became aware of a work by De Roha and Schepberg ([12]), in which a connection is established between the Clifford and Jordan-Wigner algebras, equivalent to the correspondence introduced above between (3) and (6).

transformations of one or another subgroup of the automorphism group of the basic algebra of real spinors is completely determined by the law of transformation of the matrices B_σ, B^σ, connected by relations (6). In other words, if the matrices B_σ, B^σ are transformed into new matrices $\widetilde{B}_\sigma, \widetilde{B}^\sigma$, connected by the same relations, then the transformed spinor ψ will be determined by formula (8), in which B_σ and ψ_0 are replaced by $\widetilde{B}_\sigma$ and $\widetilde{\psi}_0$, while all coefficients $\psi^{[\sigma_1 \cdots \sigma_k]}$ remain the same.

If $\psi^{[\sigma_1 \cdots \sigma_k]}$ are complex numbers, then in this case we shall call ψ a "real spinor" over the field of complex numbers, in order to emphasize that the transformation law of ψ is determined by the automorphism group of precisely the abstract basic algebra of real spinors, and not by the general algebra of alternions (9). If in expression (8) the coefficients $\psi^{[\sigma_1 \cdots \sigma_k]}$ with odd or even values of k are equal to zero, then, in agreement with the definition from (10), real spinors with this restriction will be called real semispinors of the 1st and 2nd kind.

We shall consider the set of columns of the form (8) as a left $\{B \times P\}$-module. According to known results in ring theory (4), this module is isomorphic to the factor-algebra of the algebra $\{B \times P\}$ by the left ideal $(0 : \psi_0)$, formed by the set of elements of $\{B \times P\}$ that turn ψ_0 into zero.

Theorem 4. The factor-algebra $\{B \times P\} - (0 : \psi_0)$ is isomorphic to the Grassmann algebra over the field P. Moreover, to each antisymmetric tensor with components $\psi^{[\sigma_1 \cdots \sigma_k]}$ there corresponds the element

$$\frac{1}{k!}\psi^{[\sigma_1 \cdots \sigma_k]} B_{\sigma_1} \cdots B_{\sigma_k}$$

of this factor-algebra, and to the exterior product of antisymmetric tensors with components $\psi^{[\sigma_1 \cdots \sigma_k]}$ and $\varphi^{[\tau_1 \cdots \tau_s]}$ there corresponds the product

$$\frac{1}{k!}\psi^{[\sigma_1 \cdots \sigma_k]} B_{\sigma_1} \cdots B_{\sigma_k}$$

by

$$\frac{1}{s!}\varphi^{[\tau_1 \cdots \tau_s]} B_{\tau_1} \cdots B_{\tau_s}.$$

Since second-quantized wave functions of a fermion field, according to Kastler (2) (see also (3)), may be regarded as elements of a certain Grassmann algebra, Theorem 4 makes it possible to establish a correspondence between multidimensional real spinors and the states of a fermion field. In order to pass to the description of a joint fermion-boson field, we introduce, for operators with relations (2), the concrete representation $C^\mu = \partial/\partial q_\mu$, $C_\mu = q_\mu$, and, assuming that $B^\sigma \psi_0 = 0$, $C^\mu \psi = 0$, put

$$\psi = \left(\sum \frac{1}{k!\,p!}\psi^{[\sigma_1\cdots\sigma_k](\mu_1\cdots\mu_p)} B_{\sigma_1}\cdots B_{\sigma_k} C_{\mu_1}\cdots C_{\mu_p}\right)\psi_0, \tag{9}$$

where for $k = 0$ or $p = 0$ $B_{\sigma_1}\cdots B_{\sigma_k}$ or, respectively, $C_{\mu_1}\cdots C_{\mu_p}$ are taken to be equal to the identity operator.

Definition 5. A column ψ, expressed in terms of the basic real spinor by formula (9), will be called a **generalized real spinor** if the law of transformation of ψ under transformations of one or another subgroup of the automorphism group of the algebra $\{B \times C \times P\}$ is completely determined by the law of transformation of the matrices B_σ, B^σ and the operators C_μ, C^μ, connected by relations of type (1) and (2).

The last and decisive stage in the construction of an algebraic model of quantized fields is the establishment of a connection with representations of the fundamental Lie group G with structural constants $c^{\lambda_3}_{\lambda_1\lambda_2}$, characterizing the invariant properties of physical phenomena (11). Let Y_λ

and Z_λ are infinitesimal operators of two representations of the group G, characterized by matrices with elements $(Y_\lambda)^{\sigma_1}_{\sigma_2}$ and $(Z_\lambda)^{\mu_1}_{\mu_2}$. Under continuous transformations from G, ψ^{σ_1} and ψ^{μ_1} pass into

$$\widetilde{\psi}^{\sigma_1} = \left(e^{\xi^\lambda Y_\lambda}\right)^{\sigma_1}_{\sigma_2}\psi^{\sigma_2}, \qquad \widetilde{\psi}^{\mu_1} = \left(e^{\xi^\lambda Z_\lambda}\right)^{\mu_1}_{\mu_2}\psi^{\mu_2},$$

and, correspondingly, the transformation law of the tensors

$$\psi^{[\sigma_1\cdots](\cdots\mu_p)}$$

is defined. From permutation relations of the type (1) and (2) there follows

Theorem 5. If the matrices Y_λ and Z_λ are related by the relations

$$[Y_{\lambda_1}Y_{\lambda_2}] = c^{\lambda_3}_{\lambda_1\lambda_2}Y_{\lambda_3}, \qquad [Z_{\lambda_1}Z_{\lambda_2}] = c^{\lambda_3}_{\lambda_1\lambda_2}Z_{\lambda_3},$$

then the operators

$$\hat{Y}_\lambda = (Y_\lambda)^{\sigma_1}_{\sigma_2}B_{\sigma_1}B^{\sigma_2} \quad \text{and} \quad \hat{Z}_\lambda = (Z_\lambda)^{\mu_1}_{\mu_2}C_{\mu_1}C^{\mu_2}$$

are related by the same relations. If, further,

$$\hat{X}_\lambda = \hat{Y}_\lambda + \hat{Z}_\lambda,$$

then

$$[\hat{X}_\lambda, B_\sigma] = (Y_\lambda)^{\sigma_1}_{\sigma}B_{\sigma_1}, \qquad [\hat{X}_\lambda, B^\sigma] = (Y_\lambda)^{\sigma}_{\sigma_1}B^{\sigma_1},$$
$$[\hat{X}_\lambda, C_\mu] = (Z_\lambda)^{\mu_1}_{\mu}C_{\mu_1}, \qquad [\hat{X}_\lambda, C^\mu] = (Z_\lambda)^{\mu}_{\mu_1}C^{\mu_1}.$$

Let us now choose a subgroup of the automorphism group of the algebra $\{B \times C \times P\}$, which is mentioned in Definition 5, in such a way that continuous transformations from this subgroup are characterized by operators

$$\hat{T} = e^{\xi^\lambda \hat{X}_\lambda},$$

which take $B_\sigma, \dots, C^\mu$ into

$$\widetilde{B}_\sigma = \hat{T} B_\sigma \hat{T}^{-1}, \dots, \quad \widetilde{C}^\mu = \hat{T} C^\mu \hat{T}^{-1}.$$

Putting

$$|\psi\rangle = \left(\sum \frac{1}{k!\,p!} \psi^{[\sigma_1 \dots \sigma_k](\mu_1 \dots \mu_p)} B_{\sigma_1} \dots C_{\mu_p} \right) \psi_0,$$

$$\langle\varphi| = \psi_0 \left(\sum \frac{1}{k!\,p!} \varphi_{[\sigma_1 \dots \sigma_k](\mu_1 \dots \mu_p)} B^{\sigma_1} \dots C^{\mu_p} \right),$$

where

$$\varphi_{[\sigma_1 \dots \sigma_k](\mu_1 \dots \mu_p)}$$

transform dually with respect to

$$\psi^{[\sigma_1 \dots \sigma_k](\mu_1 \dots \mu_p)},$$

from Theorem 5, taking into account $\hat{T}\psi_0 = \psi_0$, we obtain that the transformed $|\psi\rangle$ and $\langle\varphi|$ can be obtained from the original ones as a result of replacing $B_\sigma, \dots, C^\mu$ by $\widetilde{B}_\sigma, \dots, \widetilde{C}^\mu$, or as a result of replacing

$$\psi^{[\sigma_1 \dots](\dots \mu_p)} \quad \text{and} \quad \varphi_{[\sigma_1 \dots](\dots \mu_p)}$$

by

$$\widetilde{\psi}^{[\sigma_1 \dots](\dots \mu_p)} \quad \text{and} \quad \widetilde{\varphi}_{[\sigma_1 \dots](\dots \mu_p)}.$$

The generalized real spinor $|\psi\rangle$ and the "dual" real spinor $\langle\varphi|$ may be regarded as analogues of the state amplitude and the "dual" state amplitude of the theory of quantized fields. For a complete identification it is still necessary to consider the passage to the infinite-dimensional Hilbert space corresponding to infinite-dimensional representations of the inhomogeneous Lorentz group. The model constructed also makes it possible to consider various generalizations of the usual theory of quantized fields, connected with replacing the inhomogeneous Lorentz group by other Lie groups.

In conclusion, let us emphasize that we could dispense with the use of representations of the algebra $\{b \times c \times P\}$ and construct the theory of generalized real spinors exclusively in terms of an abstract associative algebra. In particular, $|\psi\rangle$ and $\langle\varphi|$ would then have to be replaced by factor algebras with respect to abstract left and right ideals corresponding to the ideals $(0 : \psi_0)$ and $(0 : \psi_0')$. This opens possibilities for such a purely algebraic reconstruction of the existing theory of quantized fields, in which the apparatus of the theory of abstract associative rings and algebras will serve as the mathematical basis of the physical theory.

Received
15 VIII 1963

CITED LITERATURE

[1] N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Quantum Theory of Fields*, 1957.
[2] D. Kastler, *Introduction à l' électrodynamique quantique*, 1961.
[3] P. K. Rashevskii, *UMN*, **13**, 3 (1958).
[4] N. Jacobson, *Structure of Rings*, 1961.
[5] G. A. Zaitsev, *Novye knigi za rubezhom*, A, 1962, No. 12, 39 (1962) (review of [2]).
[6] G. A. Zaitsev, in: *First All-Union Geometry Conference*, 1962.
[7] G. A. Zaitsev, *ZhETF*, **25**, 653, 667 (1953).
[8] G. A. Zaitsev, *ZhETF*, **29**, 166 (1955).
[9] B. A. Rosenfeld, *Non-Euclidean Geometries*, 1955.
[10] É. Cartan, *The Theory of Spinors*, 1947.
[11] G. A. Zaitsev, in: M.-A. Tonnelat, *Les principes de la théorie électromagnétique et de la relativité*, 1962, pp. 444–475.
[12] B. A. L. de-Rocha Barros, M. Schönberg, *Rev. Union Mat. Argent. y Asoc. Fis. Argent.*, **20**, 239 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.