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Abstract

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DIFFERENCE SCHEMES FOR SOLVING THE CAUCHY PROBLEM FOR HYPERBOLIC SYMMETRIC SYSTEMS

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In the present note we give several difference schemes, convergent in the norm $\mathcal{L}_2$, for solving the Cauchy problem for a linear symmetric hyperbolic system with s independent variables and m unknown functions

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{s} A^k \frac{\partial u}{\partial x_k} + Bu = f, \qquad u(x,t)\big|_{t=0} = u_0(x). \tag{1}$$

Here $x = (x_1, \dots, x_s)$ is a point of the s-dimensional real space R_s, $-\infty < x_k < +\infty$, $0 \le t \le T$; $u(x,t)$, $f(x,t)$ are vector-functions with values in m-dimensional real space; $A^k(x,t)$, $B(x,t)$ are real symmetric matrices $m \times m$. With respect to the initial data it is assumed that $u_0(x) \in \mathcal{L}_2(-\infty, +\infty)$.

Let us make some preliminary remarks. By the space $\mathcal{L}_2(-\infty, +\infty)$ we mean the space of all square-integrable in R_s m-dimensional vector-functions $u, v, \dots$, in which the scalar product and the norm are defined as follows:

$$(u,v) = \int_{R_s} \sum_{i=1}^{m} u^{(i)} v^{(i)}\, dx = \int_{R_s} u' v\, dx, \qquad \|u\|^2 = (u,u).$$

The prime denotes transposition. For a linear operator C,

$$\|C\| = \sup_{\|u\|=1} \|Cu\|.$$

By virtue of the hyperbolicity and symmetry of the system, the Cauchy problem is well posed in $\mathcal{L}_2(-\infty, +\infty)$. Since A^k are real symmetric matrices, they are orthogonally similar to diagonal matrices with real functions on the diagonals, i.e. $A^k = (a^k)^{-1} \lambda^k a^k$; $a^k(x,t)$ are orthogonal matrices composed of the coordinates of the left eigenvectors of the matrices A^k; $\lambda^k(x,t)$ are diagonal matrices composed of the corresponding eigenvalues. Then problem (1) may be written in the form

$$\frac{\partial u}{\partial t}+\sum_1^s A_1^k\frac{\partial u}{\partial x_k}+\sum_1^s A_2^k\frac{\partial u}{\partial x_k}+Bu=f,\qquad u(x,t)\big|_{t=0}=u_0(x),$$

$$A_\alpha^k=(a^k)^{-1}\lambda_\alpha^k a^k,\qquad \alpha=1,2,\qquad \lambda_1^k\geqslant 0,\quad \lambda_2^k\leqslant 0$$

(a symmetric matrix $A\geqslant 0$ ($\leqslant 0$) if all its eigenvalues are $\geqslant 0$ ($\leqslant 0$)).

In the space x,t we choose a rectangular grid with time step $\tau=T/N$ (N is a positive integer) and with step h_k in space–

variable x_k. We introduce the notation

$$T_{\pm k}u(x_1,\dots,x_k,\dots,x_s,t)=u(x_1,\dots,x_k\pm h_k,\dots,x_s,t),$$

$$u^n(x)=u(x,n\tau),\qquad n\tau=t,$$

$$h_k^{-1}(T_k-E)u=D_ku,\qquad h_k^{-1}(E-T_{-k})u=D_{-k}u.$$

Consider the following difference approximations of problem (1).

Difference scheme 1.

$$u^{n+1}=C^nu^n+\tau(-B^nu^n+f^n),\qquad u^0=u_0(x),$$

$$C^n=\left[I-\sum_1^s\varkappa_k(A_1^k-A_2^k)^n\right]E+\sum_1^s\varkappa_k(A_1^k)^nT_{-k}-\sum_1^s\varkappa_k(A_2^k)^nT_k;\qquad (2)$$

I is the identity matrix; E is the identity transformation operator; $\varkappa_k=\tau/h_k$. Scheme (2) approximates problem (1) and has first order of accuracy. If the matrices A_α^k satisfy the Lipschitz condition in x, i.e.

$$(T_{\pm k}-E)A_\alpha^k=O(h_k)\quad\text{and}\quad I-\sum_1^s\varkappa_k(A_1^k-A_2^k)\geqslant 0\quad\text{for any } t=n\tau,\ 0\leqslant t\leqslant T,$$

then the difference operator C^n has positive symmetric matrices as coefficients. According to the Friedrichs criterion (1), the norm of such an operator in $\mathcal{L}_2$ is $1+O(\tau)$ for all $0\leqslant t=n\tau\leqslant T$, whence the stability of scheme (2) follows under the condition that $\|B^n\|<\infty$, $0\leqslant t=n\tau\leqslant T$. By the equivalence theorem, a stable and locally accurate difference scheme converges. Consequently, scheme (2) converges in $\mathcal{L}_2$.

Difference scheme 2.

$$u^{n+1} = C_s^n C_{s-1}^n \dots C_1^n u^n + \tau(-B^n u^n + f^n), \qquad u^0 = u_0(x),$$

$$C_k^n = \left[I - \varkappa_k (A_1^k - A_2^k)^n\right] E + \varkappa_k (A_1^k)^n T_{-k} - \varkappa_k (A_2^k)^n T_k, \qquad k = 1, 2, \dots, s. \quad (3)$$

This scheme is equivalent to the scheme with fractional steps

$$u^{n+k/s} = C_k^n u^{n+(k-1)/s}, \qquad u^{n+1} = C_s^n u^{n+(s-1)/s} + \tau(-B^n u^n + f^n), \qquad u^0 = u_0(x), \quad (3')$$

where $u^{n+k/s}$, $k = 1, 2, \dots, s-1$, are auxiliary functions.

It is not difficult to verify that these schemes approximate problem (1). Under the restriction $I - \varkappa_k (A_1^k - A_2^k) \geqslant 0$ on $\varkappa_k$, or

$$\varkappa_k \leqslant \left[\sup_{(k,t)} |\lambda_{\max}(A^k)|\right]^{-1}, \quad (4)$$

the operators C_k^n satisfy the Friedrichs criterion, and therefore schemes (3) and (3′) are stable in $\mathcal{L}_2$, i.e. they converge.

Scheme (3) can be obtained in the following way (see (2)). Considering s auxiliary systems

$$\frac{\partial u^{(k)}}{\partial t} + A^k \frac{\partial u^{(k)}}{\partial x_k} = 0, \qquad k = 1, 2, \dots, s,$$

reducing them to characteristic form (which is possible by virtue of the hyperbolicity of system (1)) and replacing the spatial derivatives by "forward" or "backward" difference quotients (see (3)) depending on the slope of the charac-

characteristics at the point x, $t = n\tau$, we obtain "one-dimensional" difference schemes

$$(a^k)^n [u^{(k)}]^{n+1} = \{[a^k - \varkappa_k (\lambda_1^k - \lambda_2^k) a^k]^n E + \varkappa_k (\lambda_1^k a^k)^n T_{-k} - \varkappa_k (\lambda_2^k a^k)^n T_k\} (u^{(k)})^n. \quad (5)$$

Multiplying (5) on the left by $(a^k)^{-1}$, we have

$$[u^{(k)}]^{n+1} = C_k^n [u^{(k)}]^n, \tag{6}$$

where C_k^n are determined from (3). The difference scheme for the multidimensional problem (1) can be obtained in the form of the "product" of schemes (6)

$$u^{n+1} = C_s^n \cdots C_1^n u^n + (-\tau B^n u^n + f^n), \qquad u^0 = u_0(x),$$

and coincides with scheme (3).

Difference scheme 3.

$$u^{n+1} = R_s^n \cdots R_1^n u^n + \tau f^n, \qquad u^0 = u_0(x),$$

$$R_k^n = [(a^k)^{-1}]^{n+1} C_k^n [a^k]^n, \qquad C_k^n = E - \tau(\lambda_1^k D_{-k} + \lambda_2^k D_k)^n + \tau(\mu^k)^n, \tag{7}$$

$$\mu^k = \left(\frac{\partial a^k}{\partial t} + \lambda^k \frac{\partial a^k}{\partial x_k} - a^k B^{(k)}\right)(a)^{-1}, \qquad \sum_1^s B^{(k)} = B, \qquad A^k = (a^k)^{-1}\lambda^k a^k.$$

This scheme is also obtained with the aid of s auxiliary systems

$$\frac{\partial u^{(k)}}{\partial t} + A^{(k)} \frac{\partial u^{(k)}}{\partial x_k} + B^{(k)} u^{(k)} = 0, \qquad k = 1, 2, \dots, s. \tag{8}$$

If systems (8) are reduced to characteristic form, the invariants $r^{(k)} = a^k u^{(k)}$ are introduced, and then the difference schemes obtained by replacing the spatial derivatives by "forward" or "backward" difference quotients, depending on the slope of the characteristics (see (3)), are taken, then for $u^{(k)}$ we obtain the difference schemes

$$[u^{(k)}]^{n+1} = R_k^n [u^{(k)}]^n, \tag{9}$$

where R_k^n have the form indicated in (7). Scheme (7) is obtained as the "product" of the one-dimensional schemes (9). It can be shown that $\|C_k^n\| = 1 + O(\tau)$, $0 \le n\tau = t \le T$, under condition (4) on $\varkappa$ and under the assumption that $\|\mu\| < \infty$ and λ^k satisfy the Lipschitz condition in x. But $\|R_k^n\| = \|C_k^n\|$ by virtue of the orthogonality of the matrices a^k. Thus, scheme (5) is stable, and the approximation is not hard to verify. By the equivalence theorem, (5) converges in $\mathcal{L}_2$.

Difference scheme 4.

$$\Lambda^n u^{n+1} = u^n + \tau f^n, \qquad u^0 = u_0(x),$$

$$\Lambda^n = (I + \tau B^n)E + \tau \sum_1^s (A_1^k D_{-k} + A_2^k D_k)^n. \tag{10}$$

Under the condition that $A_\alpha^k,\ \alpha = 1, 2$, satisfies the Lipschitz condition in x, τ is small, and $\|B\| < \infty$ for all $0 \le t = n\tau \le T$, one can show convergence of scheme (10) in $\mathcal{L}_2$; moreover, no restrictions are imposed on $\varkappa_k$.

Difference scheme 5.

$$\Lambda_1^n \Lambda_2^n \cdots \Lambda_s^n u^{n+1} = u^n + \tau f^n, \qquad u^0 = u_0(x),$$

$$\Lambda_k^n = E + \tau (A_1^k)^n D_{-k} - \tau (A_2^k)^n D_k, \qquad k = 1, 2, \dots, s-1,$$

$$\Lambda_s^n = (I + \tau B^n)E + \tau (A_1^s)^n D_{-s} - \tau (A_2^s)^n D_s$$

or

$$u^n = \Lambda_1^n u^{n+1/s} - \tau f^n, \qquad u^{n+(k-1)/s} = \Lambda_k^n u^{n+k/s}, \qquad k = 2, \dots, s, \qquad u^0 = u_0(x).$$

These equivalent schemes converge in $\mathcal{L}_2$ for arbitrary $\varkappa_k$.

Difference scheme 6.

$$u^{n+1} = C^n u^n + \tau f^n, \qquad u^0 = u_0(x), \qquad C^n = C_s^n \dots C_1^n, \qquad C_k^n = (L_k^n)^{-1} M_k^n, \tag{11}$$

$$L_k^n = [I + 0.5\,\tau B^{(k)}]^n E + \tau (A_1^k)^n D_k + \tau (A_2^k)^n D_{-k},$$

$$M_k^n = [I - 0.5\,\tau B^{(k)}]^n E - \tau (A_2^k)^n D_{-k} - \tau (A_3^k)^n D_k,$$

$$A_\alpha^k = (a^k)^{-1} \lambda_\alpha^k a^k, \qquad \sum_{\alpha=1}^4 \lambda_\alpha^k = \lambda^k.$$

The elements of the diagonal matrices $\lambda_\alpha^k,\ \alpha = 1, 2, 3, 4$, satisfy the inequalities

$$\lambda_1^k > \frac{1}{\varkappa_k}, \qquad \lambda_2^k < \frac{1}{\varkappa_k}, \qquad \lambda_3^k \geq -\frac{1}{\varkappa_k}, \qquad \lambda_4^k < -\frac{1}{\varkappa_k}.$$

Scheme (11) is equivalent to the scheme with fractional steps

$$u^{n+k/s} = C_k^n u^{n+(k-1)/s}, \qquad k = 1, 2, \dots, s-1,$$

$$u^{n+1} = C_s^n u^{n+(s-1)/s} + \tau f^n, \qquad u^0 = u_0(x), \tag{11'}$$

$u^{n+k/s},\ k = 1, 2, \dots, s-1$, are auxiliary functions.

Scheme (11) is obtained with the aid of the auxiliary systems (8), if for each of them one takes Carlson' s scheme (see ([4])) and then the product of the "one-dimensional" schemes. One can prove the convergence of scheme (11) in $\mathcal{L}_2$ for arbitrary $\varkappa_k$.

Difference schemes 1, 2, 4, 5 can be applied to the solution of the Cauchy problem in the case of the quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + \sum_1^s A^\alpha \frac{\partial u}{\partial x_\alpha} + B = 0, \qquad u(x,t)\big|_{t=0} = u_0(x),$$

$$A^k = A^k(x, t, u), \qquad B = B(x, t, u). \tag{12}$$

Schemes 1–6 can also be used for solving problem (12) in the following way. For system (12), by differentiating with respect to x_k we obtain the prolonged system

$$\frac{\partial u}{\partial t} = -\left(\sum_1^s A^k p_k + B\right), \tag{13}$$

$$\frac{\partial p_k}{\partial t} + \sum_{\alpha=1}^s A^\alpha \frac{\partial p_\alpha}{\partial x_k} = F_k, \qquad F_k = -\sum_{\alpha=1}^s \left(\frac{\partial A^\alpha}{\partial x_k} + \frac{\partial A^\alpha}{\partial u} p_k\right) p_\alpha - \left(\frac{\partial B}{\partial x_k} + \frac{\partial B}{\partial u} p_k\right), \tag{14}$$

$$k = 1, 2, \dots, s, \qquad u(x,t)\big|_{t=0} = u_0(x), \qquad p_k(x,t)\big|_{t=0} = \frac{\partial u_0}{\partial x_k},$$

which is semilinear.

Considering u known at $t = n\tau$, in order to determine the approximate value of p_k at the time $t + \tau$ one may use for (14) one of the difference schemes 1-6, and then determine u^{n+1} by integrating the system of ordinary differential equations (13).

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Note: Figure translations are in progress. See original paper for figures.