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Abstract

Full Text

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ON AN INVERSE PROBLEM FOR THE WAVE EQUATION

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In the present note a theorem is proved on the uniqueness of the solution of an inverse problem for the wave equation.

Consider the equation

$$n^2 \frac{\partial^2 u}{\partial t^2} = \Delta u, \tag{1}$$

where u is a function of the three variables x, y, t; n is a function of the variables x, y.

We shall consider the following formulation:

In the plane x, y a domain D_0 is given. The function $n(x, y) > 0$ is continuous and is identically equal to one outside D_0. In addition, in some domain D_1, $D_1 \cap D_0$ empty, a family G of solutions of (1) is given for all $t > 0$. It is required to determine the function $n(x, y)$ inside D.

We note that an analogous formulation in the case of one variable was considered in [1]; close in character to the formulation considered here is the widely known inverse Sturm–Liouville problem.

Let us formulate the uniqueness theorem for the posed problem for one family G.

Theorem. Let D_0 and D_1 be bounded simply connected domains, let D_2 also be some bounded simply connected domain not intersecting D_0, D_1, and let G be the set of solutions of (1) satisfying the following initial conditions:

$$u(x, y, 0) = 0,$$

$$\frac{\partial u(x, y, 0)}{\partial t} = \delta(x - x_0,\ y - y_0), \tag{2}$$

where $Q(x_0, y_0)$ is an arbitrary point of D_2.

Then the solution of the formulated inverse problem is unique, i.e., the function $n(x, y)$ is determined uniquely inside D_0.

We shall present the brief content of the proof of the theorem. Denote the solution of the Cauchy problem (2) for equation (1) by $u(x, y, x_0, y_0, t)$ and consider the function

$$v(x, y, x_0, y_0, \lambda) = \int_0^\infty u(x, y, x_0, y_0, t) \cos \lambda t \, dt.$$

It is easy to see:

$$\Delta v = -\delta(x - x_0,\ y - y_0) - \lambda^2 n^2 v. \tag{3}$$

The function v is the fundamental solution of the Helmholtz equation

$$\Delta v = -\lambda^2 n^2 v$$

with a singularity at the point $Q(x_0, y_0)$. As is known, the fundamental solutions of elliptic equations with analytic coefficients are analytic functions both of the independent variables x, y and of the coordinates of the singularity x_0, y_0 everywhere outside the singularity.

By the conditions of the theorem the function v is given in the domain D_{12} of the four-dimensional space $R(x, y, x_0, y_0)$, which is the direct product of the domains D_1, D_2 of the spaces $P(x, y), Q(x_0, y_0)$ ($R \in D_{12}$ if $P \in D_1,\ Q \in D_2$). The function $n(x, y)$ is identically equal to one outside D_0. Consequently, by virtue of the uniqueness of analytic continuation, the function v may be considered given everywhere outside the domain D_{00}—the product of D_0 by itself ($R \in D_{00}$, if $P \in D_0;\ Q \in D_0$).

Let D_3 be some bounded domain containing the domains D_0, D_1, D_2, and let

$$r^2 = (x - \xi)^2 + (y - \eta)^2, \qquad r_0^2 = (x_0 - \xi)^2 + (y_0 - \eta)^2.$$

It follows from (3) that for $P \in D_3$:

$$v(x, y, x_0, y_0, \lambda) = \frac{1}{2\pi} \ln[(x - x_0)^2 + (y - y_0)^2]$$

$$-\frac{1}{2\pi} \lambda n^2(x, y) \int_{D_0} v(\xi, \eta, x_0, y_0, \lambda) \ln r \, d\xi \, d\eta + \tilde{v}(x, y, x_0, y_0, \lambda), \tag{4}$$

$$\tilde{v} = \frac{1}{2\pi} \int_{\Gamma_3} \left(v \frac{\partial}{\partial n} \ln r - \frac{\partial v}{\partial n} \ln r \right) ds,$$

where Γ_3 is the boundary of D_3.

Denote by $v_1(x, y, x_0, y_0)$ the function

$$v_1 = \frac{\partial}{\partial\lambda}\left[\frac{\partial^2 v(x,y,x_0,y_0,0)}{\partial x\,\partial x_0} + \frac{\partial^2 v(x,y,x_0,y_0,0)}{\partial y\,\partial y_0}\right].$$

It is not difficult to show that, for $R \in D_{33}$ (D_{33} is the product of D_3 by itself), the function v_1 is equal to

$$v_1 = \int_{D_0} n(\xi,\eta)\frac{(x-\xi)(x_0-\xi)+(y-\eta)(y_0-\eta)}{r^2 r_0^2}\,d\xi\,d\eta + \tilde{v}_1, \qquad (5)$$

where

$$\tilde{v}_1(x,y,x_0,y_0) = \frac{\partial}{\partial\lambda}\left[\frac{\partial^2 \tilde{v}}{\partial x\,\partial x_0} + \frac{\partial^2 \tilde{v}}{\partial y\,\partial y_0}\right].$$

In view of (5), the functions $v_1, \tilde{v}_1$ for $R \in D_{33}$, $R \notin D_{00}$ (D_{33} is the product of D_3 by itself) are analytic functions of the variables x, y, x_0, y_0.

Putting in (5)

$$x = x_0, \qquad y = y_0; \qquad P(x,y) \in D_1,$$

we obtain

$$v_2 = v_1(x,y,x,y) - \tilde{v}_1(x,y,x,y) = \int_{D_0} n(\xi,\eta)\frac{1}{r^2}\,d\xi\,d\eta. \qquad (6)$$

Let us now consider the function

$$w(x,y,x_1,y_1) = \int_{D_0} n(\xi,\eta)\frac{1}{\rho^2}\,d\xi\,d\eta,$$

where

$$\rho^2 = (x-\xi)^2 + (y-\eta)^2 + x_1^2 + y_1^2.$$

The function w is the potential of a simple layer in the four-dimensional space $R_1(x,y,x_1,y_1)$ with density $n(x,y)$, distributed on the two-dimensional manifold:

$$P(x,y) \in D_0; \qquad x_1 = y_1 = 0 \qquad (D_0').$$

In view of the foregoing, the function w may be regarded as given on the two-dimensional manifold:

$$P(x,y) \in D_1; \qquad x_1 = y_1 = 0 \qquad (D_1^1)$$

It can be shown that the function w is uniquely determined in the entire space R_1 by its values in D_1^1, whence, by the known theorems of potential theory, the assertion of the theorem follows.

We note that the theorem generalizes to the case of the wave equation in a space of any number of dimensions, to the heat equation, and also to certain equations of hyperbolic and elliptic type of higher orders.

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CITED LITERATURE

1. M. G. Krein, *Dokl. Akad. Nauk SSSR*, **82**, 669 (1959).

Note: Figure translations are in progress. See original paper for figures.