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Abstract

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ON UNIFORM STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS

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Here we shall prove a theorem which gives an exhaustive solution to the question of the so-called uniform stabilization of the solution of the Cauchy problem for a Petrovskii-parabolic system whose coefficients depend only on time t.

Consider the system

$$\frac{\partial u}{\partial t} = \sum_{|k|=2b} A_k D^k u. \tag{1}$$

Here D^k denotes $(-i)^k \dfrac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$, $k := k_1 + k_2 + \dots + k_n$; A_k is a square matrix with constant elements. S. D. Eidelman showed in ([1]) that for Green' s matrix of (1)

$$G(t,x) = \frac{1}{(2\pi)^n} \int \exp\left[i(x,\sigma) - t \sum_{|k|=2b} A_k \sigma_1^{k_1} \sigma_2^{k_2} \cdots \sigma_n^{k_n}\right] d\sigma \tag{2}$$

the estimate

$$\left| \frac{\partial^k G(t,x)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| < c_1 t^{-(n+k)/2b} \exp\{-c_2(|x| t^{-1/2b})^q\}, \tag{3}$$

is valid,

$$(x,\sigma) = x_1\sigma_1 + x_2\sigma_2 + \dots + x_n\sigma_n, \quad |x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}, \quad q = \frac{2b}{2b-1}.$$

It is known that, in the uniqueness class, the solution of problem (1), constructed from a bounded measurable initial vector-function

$$u(0,x) = u_0(x), \tag{4}$$

is given by the Poisson integral:

$$u(t,x) = \int G(t, x-\xi) u_0(\xi)\, d\xi \quad (d\xi = d\xi_1\, d\xi_2 \dots d\xi_n). \tag{5}$$

From the fact that the identity matrix E satisfies system (1), it follows that

$$\int G(t, x-\xi)\, d\xi = E.$$

Before formulating the main result, let us introduce some definitions.

Definition 1. We shall say that a bounded measurable vector-function $u_0(x)$ has a **uniform limiting mean** equal to l ($l = (l_1, l_2, \dots, l_n)$), if for any $\varepsilon > 0$ there is an $N_0(\varepsilon)$ such that for every cube V_N^η with side equal to $2N$ and center at the point $\eta(\eta_1, \eta_2, \dots, \eta_n)$

$$\left| \frac{1}{(2N)^n} \int_{V_N^\eta} u_0(x)\, dx - l \right| < \varepsilon$$

for all η and all $N > N_0$.

Definition 2. A vector function $u(t,x)$ is said to be **uniformly stabilized** to l if, for every $\varepsilon > 0$, one can indicate a $T_0(\varepsilon)$, independent of x, such that for $t > T_0$

$$|u(t,x) - l| < \varepsilon.$$

Theorem. *For the uniform stabilization of the solution of problem (1), (4), represented by the Poisson integral, it is necessary and sufficient that the initial vector function $u_0(x)$ have a uniform limiting mean.*

Proof. Necessity. For simplicity we shall assume that the moduli of the components of the initial vector function $u_0(x)$ are bounded by one ($|u_0^j| < 1,\ j = 1, 2, \dots, m$) and that l is the zero vector. We shall prove by contradiction.

Let

$$u(t,x) = \int G(t, x-\xi) u_0(\xi)\, d\xi \longrightarrow 0 \quad \text{as } t \to \infty$$

uniformly in x, while the initial vector function has no uniform limiting mean. The latter means that, for some $\varepsilon_0 > 0$ and any positive N_k ($k = 1, 2, \dots$), there exist $N > N_k$ and points $\eta^k(\eta_1^k, \eta_2^k, \dots, \eta_n^k)$ such that, for at least one of the components of the vector function $u_0(x)$ (for definiteness, $u_0^1(x)$),

$$\left| S_1(N_k, \eta^k) \right| = \left| \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} u_0^1(\xi)\, d\xi \right| > \varepsilon_0. \tag{6}$$

We shall show that it follows from (6) that $u_1(t,x)$ does not tend to zero as $t \to \infty$. Indeed, choose the sequence of times

$$t_k = \left(\frac{\varepsilon_0 N_k}{4b^{n+1}mn} \right)^{2b}$$

(N_k is some sequence tending to infinity, and B is a constant which will be chosen below) and average $u_1(t,x)$ in the following way:

$$\left| \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} u_1(t_k, x)\, dx \right| =$$

$$= \left| \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} \int_{E_n} \sum_{j=1}^{m} G_{1j}(t, x-\xi) [u_0^j(x) + u_0^j(\xi) - u_0^j(x)]\, d\xi\, dx \right| =$$

$$= \Bigg| S_1(N_k, \eta^k) + t_k^{n/2b} \int_{V_B^0} \sum_{j=1}^{m} G_{1j}(t_k, y t_k^{1/2b})\, dy \, \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} [u_0^j(x - t_k^{1/2b} y) - u_0^j(x)]\, dx$$

$$+ t_k^{n/2b} \int_{E_n - V_B^0} \sum_{j=1}^{m} G_{1j}(t_k, y t_k^{1/2b})\, dy \, \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} [u_0^j(x - t_k^{1/2b} y) - u_0^j(x)]\, dx \Bigg| =$$

$$= |S_1 + I_1 + I_2| \geq |S_1| - |I_1| - |I_2|.$$

We now choose B so that

$$t_k^{n/2b} \int_{E_n - V_B^0} \sum_{j=1}^{m} \left| G_{1j}(t_k, y t_k^{1/2b}) \right| dy < \frac{\varepsilon_0}{8}.$$

This can be done by using estimate (3). Then, obviously, $|I_2| < \varepsilon_0/4$. It is asserted that $|I_1| < \varepsilon_0/4$.

Indeed, since $u_0^j(x - t^{1/2b} y)$ differs from $u_0^j(x)$ on a set whose measure is no greater than $2^n n N_k^{n-k} B t^{1/2b}$, we have

$$\left| \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} [u_0^j(x - t_k^{1/2b} y) - u_0^j(x)]\, dx \right| \leq \frac{nBt_k^{1/2b}}{N_k}.$$

Since

$$t_k^{n/2b} \int_{V_B^0} \sum_{i=1}^{m} \left| G_{1j}(t_k, y t_k^{1/2b}) \right| dy < mB^n,$$

the desired smallness of $|I_1|$ is proved.

Thus it has been proved that, for some $\varepsilon_0 > 0$, there exists a sequence of times t_k tending to infinity and a sequence of points $\eta^k(\eta_1^k, \eta_2^k, \dots, \eta_n^k)$ such that

$$\left| \frac{1}{(2N_k)^n} \int_{V_{N_k}^{\eta^k}} u_1(t_k, x)\, dx \right| > \frac{\varepsilon_0}{2}.$$

It follows from this that, for any arbitrarily large time T_0, one can indicate such $t_k > T_0$ and such points η_0^k, lying inside $V_{N_k}^{\eta^k}$, that

$$|u(t_k, \eta_0^k)| > \varepsilon_0/2.$$

The necessity is thereby proved.

Sufficiency*. Let the initial vector-function $u_0(x)$ have a uniform limiting mean equal to the zero vector.

We shall show that for every $\varepsilon > 0$ one can indicate a T_1, independent of x, such that for $t > T_1$,

$$|u(t,x)| < \varepsilon.$$

For this purpose write $u(t,x)$ in the form

$$\begin{aligned} u(t,x) &= \int G(t,y) \frac{\partial^n}{\partial y_1 \dots \partial y_n} \int_0^{y_1} \cdots \int_0^{y_n} u_0(x-\eta)\, d\eta\, dy \\ &= \int \frac{\partial^n G(t,y)}{\partial y_1 \dots \partial y_n} \int_0^{y_1} \cdots \int_0^{y_n} u_0(x-\eta)\, d\eta\, dy, \end{aligned} \tag{7}$$

since the remaining terms on the right-hand side of (7) are equal to zero by virtue of estimate (3).

Proceeding to estimate $u(t,x)$, represent the latter as the sum of two integrals, then use estimate (3) and make the change of variable $y = z\tau$ (where τ denotes $t^{1/2b}$):

$$\begin{aligned} |u(t,x)| = &\left| \int_{|y_1|\le\alpha\tau} \cdots \int_{|y_n|\le\alpha\tau} \frac{\partial^n G(t,y)}{\partial y_1 \dots \partial y_n} \int_0^{y_1} \cdots \int_0^{y_n} u_0(x-\eta)\, d\eta\, dy \right. \\ &\left. + \int_{|y_1|>\alpha\tau} \cdots \int_{|y_n|>\alpha\tau} \frac{\partial^n G(t,y)}{\partial y_1 \dots \partial y_n} \int_0^{y_1} \cdots \int_0^{y_n} u_0(x-\eta)\, d\eta\, dz \right| \\ &< c_1 \int_{|z_1|\le\alpha} \cdots \int_{|z_n|\le\alpha} \exp\{-c_2|z|^q\} \left| \frac{1}{\tau^n} \int_0^{z_1\tau} \cdots \int_0^{z_n\tau} u_0(x-\eta)\, d\eta\, dz \right| \\ &+ c_1 \int_{|z_1|>\alpha} \cdots \int_{|z_n|>\alpha} \exp\{-c_2|z|^q\} \left| \frac{1}{\tau^n} \int_0^{z_1\tau} \cdots \int_0^{z_n\tau} u_0(x-\eta)\, d\eta\, dz \right| = I_3 + I_4. \end{aligned}$$

* This part of the theorem follows from the general stabilization theorem of F. O. Porper and S. D. Eidelman ([3]); for completeness of exposition we give its proof.

Noting that

$$I_3 = c_1 \int_{|z_1|\le a} \cdots \int_{|z_n|\le a} \exp\{-c_2|z|^q\} |z_1 z_2 \cdots z_n| \, S(z_1\tau, \ldots$$

$$\ldots, z_n\tau, x)\, dz$$

(where $S(z_1\tau, \ldots, z_n\tau; x)$ is the mean of the function $u_0(x)$ over the parallelepiped of dimensions $\tau(z_1 \times z_2 \times \cdots \times z_n)$ with center at the point x), we choose a so small that $I_3 < \varepsilon/2$.

To complete the proof it remains to note that, by virtue of the assumption on the existence of a zero uniform limiting mean of $u_0(x)$, one can indicate such a T_1 that for all $t > T_1$

$$S(\tau a, x) < \frac{\varepsilon}{2} \left\{ c_1 \int \exp[-c_2|z|^q] \, |z_1 z_2 \cdots z_n| \, dz \right\}^{-1},$$

therefore $I_4 < \varepsilon/2$, $|u(t,x)| < \varepsilon$. The theorem is completely proved.

Remark. The proof given remains valid for an arbitrary Petrovsky-parabolic system of the form

$$\frac{\partial u}{\partial t} = \sum_{1\le|k|\le 2b} A_k(t) D^k u,$$

for whose Green matrix the estimate

$$D^{mG}(t,0,x) \le c_m a(t)^{-m-n} \exp\{-c(|x| a^{-1}(t))^q\} \qquad (a(t) \to \infty \text{ as } t \to \infty)$$

is valid.

Various sufficient conditions for the validity of the latter estimate are given in ([2]).

In conclusion I express my sincere gratitude to S. D. Eidelman for posing the problem solved here, and also for valuable suggestions given in the course of its solution.

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Note: Figure translations are in progress. See original paper for figures.