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Abstract

Full Text

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ON THE STABILITY OF POISEUILLE FLOW IN A PLANE CHANNEL

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Introduction. The question of the stability of Poiseuille flow in a plane channel reduces to the study of the following problem: for the equation

$$(D^2-\alpha^2)^2\varphi = i\alpha R\{(u-c)(D^2-\alpha^2)-u''\}\varphi; \tag{1}$$

$$D = d/dy, \qquad -1 < y < 1, \qquad u = 1-y^2,$$

$$\varphi = D\varphi = 0 \quad \text{for } y = \pm 1 \tag{2}$$

determine whether there exists an eigenvalue $c = c(\alpha, R)$ lying in the upper half-plane (for some values $\alpha, R > 0$). If such a c exists, the Poiseuille flow is unstable.

In the present note it is rigorously proved that Heisenberg's answer [2] (instability) (and subsequently that of Tollmien, Schlichting, Lin, Thomas; see the survey [1]) is correct.

Making the linear change in the complex plane

$$z = \frac{2}{u'(y_c)}(y-y_c),$$

where y_c is the root, close (for small c) to -1, of the equation $u(y_c) = 1-c$, we bring problem (1)–(2) to the form

$$(D^2-\beta^2)^2\varphi + (-i\rho^2)\{z(1-z/2)(D^2-\beta^2)+1\}\varphi = 0; \tag{3}$$

$$\varphi = D\varphi = 0 \quad \text{for } z = z_1(c) \text{ and } 2-z_1(c), \tag{4}$$

where $D = d/dz$, $\beta^2 = \alpha^2(1-c)$, $\rho^2 = 2\alpha R(1-c)$,

$$z_1(c) = 1-\frac{1}{\sqrt{1-c}} = -\frac{c}{2}(1+O(c)). \tag{5}$$

(Here and below, unless otherwise stated, the root of numbers close to positive ones is understood as the value close to the arithmetic one.)

Problem (3)–(4) is studied in the following way:

I. A certain fundamental system of equation (3) is studied, consisting of smooth "inviscid" solutions and "boundary-layer" solutions, as well as its dependence on the parameters β^2 and $\lambda^2 = -i\rho^2$. In view of the fact that the coefficients of (3) are even with respect to the point $z = 1$, one may seek an "even" fundamental system of solutions (consisting of functions even and odd with respect to $z = 1$). Then, for the even smooth and boundary-layer solutions, the boundary condition at $z = 2 - z_1(c)$ is satisfied automatically.

II. For this pair of functions a characteristic determinant is written down, containing the values of the functions and of their derivatives at a point z close to the origin of coordinates. Approximate values $z = z_0(\beta, \lambda)$ are found which make this determinant vanish.

III. The error is estimated; more precisely, it is shown that

$$|z_0(\beta,\lambda) - z_{\text{true}}(\beta,\lambda)| < |\operatorname{Im} z_0(\beta,\lambda)|, \tag{6}$$

i.e., the sign of the imaginary part of the true root coincides with the sign of the imaginary part of the approximate root.

IV. Finally, the equation $z = z(\beta(a,z), \lambda(aR,z))$ is investigated, and it is shown that it has a solution in the same domain of z (i.e., with $\operatorname{Im} z < 0$).

All considerations are carried out in the following ranges of variation of the parameters:

1) $|\rho| \to \infty, \quad |\arg\rho| < \varepsilon;$

2) $|\beta| \to \infty, \quad |\arg\beta| < \varepsilon;$

3) $|\lambda|^{-2/3} \ll |z| \ll 1, \quad |\arg z - \pi k| < \varepsilon;$

4) $|\beta|^{-3} \ll |\lambda| \ll |\beta|^{-5},$

where ε is a fixed small number.

1. Fundamental system for (3)

Equation (3) belongs to the class of equations studied by Wasow in his works (3), and has a fundamental system of solutions of the following form (in the domain $1 \geq |z| \gg |\rho|^{-2/3}$, $|\arg z - \pi k| < \varepsilon$):

a) A pair of "smooth solutions"(close to the solutions of the inviscid equation):

$$\varphi_1^{(m)}(z,\lambda) = \varphi_1^{(m)}(z,\infty) + O(\lambda^{-2}); \tag{7}$$

$$\varphi_2^{(m)}(z,\lambda) = \varphi_2^{(m)}(z,\infty) + R_m \quad (m = 0,1,2,3). \tag{8}$$

Here $\varphi_1(z,\infty) = \varphi_1(z)$ is the regular solution of the inviscid equation, $\varphi_2(z,\infty) = \varphi_2(z)$ is a certain special branch of the singular solution (more will be said about these functions below), and

$$R_0 = O(\lambda^{-2}z^{-2}), \qquad R_1 = O(\lambda^{-4/3}z^{-2}), \qquad R_2 = O(z^{-1}), \qquad R_3 = O(z^{-2}), \tag{9}$$

$$R_m = O(\lambda^{-2}) \quad \text{for } |z| > a.$$

b) A pair of "boundary-layer" solutions:

$$\varphi_3^{(m)}(z,\lambda) = \chi(z)(-z)^{-5/4}\left[e^{2/3\lambda(-z)^{3/2}}\right]^m\left(1 + O(\lambda^{-1}z^{-3/2})\right); \tag{10}$$

$$\varphi_4^{(m)}(z,\lambda) = \chi(z)(-z)^{-5/4}\left[e^{2/3\lambda(-z)^{3/2}}\right]^{(m)}\left(1 + O(\lambda^{-1}z^{-3/2})\right). \tag{11}$$

Here $\lambda = \rho\sqrt{-i} = \rho e^{-\pi i/4}$; $\chi(z)$ is a smooth function of z; $\chi(0) = 1$. The values of $(-z)^{3/2}$ for φ_3 and φ_4 are taken with opposite signs in such a way that $\varphi_3(z,\lambda)$ has an exponential decreasing from left to right, while $\varphi_4(z,\lambda)$ has an increasing one.

Let us show that there exist two even linearly independent solutions—a smooth one and a boundary-layer one. The solution of the Cauchy problem with data at the point $z = 1$: $\Psi(1) = \varphi_3(1)$, $\Psi'(1) = \Psi'''(1) = 0$, $\Psi''(1) = \varphi_3''(1)$, is the function

$$\Psi(z,\lambda) = C_1\varphi_3(1)\varphi_1(z,\lambda) + C_2\varphi_3(1)\varphi_2(z,\lambda) +$$

$$+ \left(1 + O(\lambda^{-1})\right)\left(\varphi_3(z,\lambda) + \varphi_3^2(1)\varphi_4(z,\lambda)\right), \tag{12}$$

where C_1 and C_2 are bounded functions of λ. For small z it has the same asymptotics as $\varphi_3(z,\lambda)$.

The even smooth solution $\Phi(z,\lambda)$ has the form

$$\Phi(z,\lambda) = C_1'(\lambda)\varphi_1(z,\lambda) + C_2'(\lambda)\varphi_2(z,\lambda) + O(\lambda^{-4})\varphi_3(1)\varphi_4(z,\lambda), \tag{13}$$

and is obtained from the solution of the Cauchy problem with data $(\Phi(1), 0, \Phi''(1), 0)$ by subtracting the even boundary layer $\Psi(z, \lambda)$ (here $\Phi(z)$ is the even solution of the inviscid equation). Our aim is to study the roots z near zero of the equation

$$f(z;\beta,\lambda) = \begin{vmatrix} \Phi(z;\beta,\lambda) & \Psi(z,\beta,\lambda) \\ \Phi'(z;\beta,\lambda) & \Psi'(z;\beta,\lambda) \end{vmatrix} = 0. \tag{14}$$

2. Fundamental system of solutions of the inviscid equation

The inviscid equation

$$\{z(1-z/2)(D^2-\beta^2)+1\}\varphi = 0 \tag{15}$$

is an equation of Fuchsian type at the points $z = 0$ and $z = 2$. It has a fun-

a fundamental system consisting of a solution regular at zero,

$$\varphi_1(z) = z\sum_{k=0}^{\infty} a_k(\beta) z^k \tag{16}$$

and a singular solution

$$\varphi_2(z) = \varphi_1(z)\ln z + \chi(z); \tag{17}$$

$$\chi(z) = \sum_{k=0}^{\infty} b_k(\beta) z^k. \tag{18}$$

Here the series (16) and (18) converge in the circle $|z| < 2$, and the coefficients $a_k(\beta)$ and $b_k(\beta)$ are entire functions of β^2 and can be found in the form of series. In particular,

$$\varphi_1(z) = z\{(1-z/2) + \beta^2(z^2/6 + \ldots) + \ldots + \beta^{2n}(c_n d^{2n} + \ldots) + \ldots\},$$

$$\chi(z) = -1 + z^2 + \ldots + \beta^2(-{}^1\!/_2\, z^2 + \ldots) + \ldots + \beta^{2n}(d_n z^{2n} + \ldots) + \ldots$$

By a simple calculation it is easy to find that the solution $\Phi(z)$ of equation (15), satisfying the condition $\Phi'(1) = 0$, has the form

$$\Phi(z) = \varphi_1(z) + k(\beta)\varphi_2(z), \qquad k(\beta) = \sum_{j=1}^{\infty} k_j \beta^{2j}, \tag{19}$$

$$k_1 = -2\int_0^1 [z(1-z/2)]^2\,dz = -k, \qquad k > 0 \tag{20}$$

(all coefficients k_j can be found in the form of numerical series or repeated integrals of the form $\int_0^1 dz\, f_1(z)\int_0^z f_2(z)\cdots\int_0^1 dz\, f_n(z)$, where $f_k(z)$ are known functions of z).

In what follows, by $\varphi_2(z)$ we shall denote the branch of the singular solution obtained by considering the branch

$$\ln z = \ln|z| + i\arg z, \qquad -3\pi/2 < \arg z < \pi/2,$$

and it is precisely for this branch of $\varphi_2(z)$ that the estimates formulated above by Wasow hold.

Let us now try, for the solution $\Phi(z)$, to find a point z_1 at which $\Phi(z_1) = 0$ (we note that we have singled out the branch $\varphi_2(z)$ that is of interest to us, so that $\Phi(z)$ is a single-valued function), i.e., to solve the problem, degenerate with respect to the small parameter λ^{-1}. The first approximation as $\beta^2 \to 0$ gives us information on the location of the root z_1 of equation (14). For small z and β we have the asymptotic formula

$$\Phi(z,\beta) = z + O(z^2) + O(\beta^2 z) + k\beta^2 + O(\beta^2 z\ln z) =$$

$$= z + k\beta^2 + o(z) + o(\beta^2).$$

Hence

$$z_1 = -k\beta^2 + o(\beta^2). \tag{21}$$

We shall see below that the term written explicitly in (21) gives the first terms of the asymptotic expansion for the solution of the full viscous problem.

3. Characteristic determinant for even solutions. Substituting in (13) the values $C_1(\infty), C_2(\infty)$ from (19) and estimate (7), we obtain

$$\Phi^{(m)}(z,\lambda) = \varphi_1^{(m)}(z) + k(\beta)\varphi_2^{(m)} + O(\lambda^{-2}) + O(\beta R_m). \tag{22}$$

Substituting the functions (12) and (22) into (14), we obtain

$$\varphi_3(z,\lambda)\left\{\left(\frac{\varphi_3'}{\varphi_3} + O(\lambda^{-1}z^{-3/2})\right)(\varphi_1(z) + k(\beta)\varphi_2(z) + O(\beta\lambda^{-2}z^{-2})) - \varphi_1'(z) - k(\beta)\varphi_2'(z) + O(\beta\lambda^{-4/3}z^{-2})\right\} = 0 \tag{23}$$

We shall first find some approximate root of this equation, and then estimate the error. In the approximate equation

$$\frac{\Phi(z)}{\Phi'(z)} = \frac{\varphi_3(z)}{\varphi_3'(z)}$$

it suffices (as will become clear below) to restrict ourselves to the terms

$$\frac{\Phi(z)}{\Phi'(z)} = z + k\beta^2 + O(\beta^4 \ln z), \qquad \frac{\varphi_3}{\varphi_3'} = -\frac{1}{\rho(-z)^{3/2}}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right).$$

Taking

$$\operatorname{Re} z_0 = \operatorname{Re}(-k\beta^2) = -k \operatorname{Re} \beta^2,$$

$$\operatorname{Im} z_0 = -\operatorname{Re}\frac{\sqrt{2}}{2}\frac{1}{\rho(-z_0)^{1/2}} = -\frac{\sqrt{2}}{2} k^{-1/2} \operatorname{Re}(\rho\beta)^{-1} < 0, \tag{24}$$

we can now estimate the error, using, for example, Newton' s method (see (4)): if z_0 is an approximate root of the equation $f(z) = 0$, $|f(z_0)/f'(z_0)| < \eta$, $\max_z |f''(z)/f'(z_0)| < K$, and $h = \eta K \to 0$ with respect to some parameter, then the true root is located in the circle $|z - z_0| < \eta$ (the condition is not necessary). Since we are interested in the sign of the imaginary part $\operatorname{Im} z$, it is enough for us to prove that $\eta = o(\operatorname{Im} z_0) = o(\rho^{-1}\beta^{-1})$.

We easily obtain

$$f(z_0) = \varphi_3(z_0) o(\beta^{4-\varepsilon}) O(\beta\rho), \qquad f^{(m)}(z) = \varphi_3(z) O(\beta^m \rho^m) \quad (m = 1, 2), \tag{25}$$

$$K\eta = o(\beta^{4-\varepsilon}) O(\beta\rho) \to 0, \qquad \eta = o(\beta^{4-\varepsilon}) = o(\rho^{-1}\beta^{-1}) \quad \text{for} \quad \beta = o(\rho^{-1/(5-\varepsilon)}).$$

Thus, we have shown that equation (23) has a root $z = Z(\beta, \lambda)$ with negative imaginary part. Let us now show that the equation

$$z = Z(\beta(z), \lambda(z));$$

also has a solution. This follows at once from the fixed-point principle for the mapping of the circle $|z_0 - z| < C\beta^{-1}\rho^{-1}$: by means of the function Z, this circle is mapped into a circle of radius $|z - z_0| = o(\beta^{-1}\rho^{-1})$.

Remarks. 1. It is easy to see that by this method one can obtain exact asymptotics in more general cases than those we have considered; in particular, asymptotics for the branches of the neutral curve in the plane.

2. The method we use can also serve for the numerical determination of the eigenvalues $c = c(\alpha, R)$, and in this sense it is also much simpler than the Heisenberg-Lin method (1).

3. The stability problems for the boundary layer on a flat plate and other problems with loss of stability of plane-parallel flows can be considered analogously.

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Note: Figure translations are in progress. See original paper for figures.