Mathematics

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1964

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Abstract

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Mathematics

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HOMOLOGIES OF THE SPACE OF UNORIENTED LOOPS AND THEIR APPLICATION TO THE CALCULUS OF VARIATIONS IN THE LARGE

(Presented by Academician P. S. Aleksandrov on 6 XI 1963)

The problem of estimating the number of geodesic loops on a Riemannian manifold was first considered in ([1]), where it was shown that on a manifold homeomorphic to an n-dimensional sphere there exists a countable sequence of geodesic loops with beginning and end (base point) at an arbitrarily chosen point of the manifold. In ([2]) A. I. Fet proved that on an arbitrary Riemannian manifold there exists at least one geodesic loop. From Serre' s results ([3]) it follows that on an arbitrary closed manifold there exists a countable sequence of geodesic loops; moreover ([4]), from this sequence one can extract a subsequence of loops whose lengths increase monotonically, but not faster than the terms of a certain arithmetic progression. (For the case of a spherical manifold this result was obtained by L. A. Lyusternik ([5]).) Estimates of the number of geodesic loops were obtained by studying the homology of the space of **oriented** loops $\Omega(M)$ of the manifold M, but in solving problems of the calculus of variations "in the large" significantly better estimates can be obtained if the variational functional is considered on spaces of **unoriented** loops, paths, and curves (cf. ([6])).

In the present article* the homology of the space of unoriented loops of the n-dimensional sphere is completely computed, and a final estimate is obtained for the number of geodesic loops on a Riemannian manifold homeomorphic to the n-dimensional sphere. A countable sequence of series of geodesics is found; each series consists of n loops, of which only two enter the previously known sequence. The computations are carried out by the method of spectral sequences ([6–8]).

1. The space of unoriented loops.

Let M^n be a Riemannian manifold homeomorphic to an n-dimensional sphere. We take an arbitrary point $m_0 \in M^n$ as the base point. In the space $\Omega(M^n)$ of oriented loops one can define an involution

$$\Omega(M^n) \xrightarrow{\Theta} \Omega(M^n), \tag{1}$$

by setting

$$f_2(t) = (\Theta f_1)(t) = f_1(1-t) \qquad (0 \le t \le 1). \tag{2}$$

Identifying in $\Omega(M^n)$ the loops satisfying condition (2), we obtain the space $\hat{\Omega}(M^n)$ of unoriented loops of the manifold M^n. The natural mapping will be called the p-projection. A one-point loop will be denoted by O. It can be shown that the spaces $\hat{\Omega}(M^n)$ and $\hat{\Omega}(M^n)/O$ are homotopically non-equivalent (cf. ([9])), and in the space $\hat{\Omega}(M^n)$ one cannot introduce an operation of multiplication of loops.

In the subspace of rectifiable loops we define a **weak** metric

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* The results of the article were reported on 11 IV 1963 at the Topological Seminar of Moscow University and on 27 IX 1963 at the IV All-Union Topological Conference in Tashkent.

by the formula

$$r_F(v_1, v_2) = \min\left[\max_t \rho(f_1(t), f_2(t)),\ \max_t \rho(f_1(t), f_2(1-t))\right], \tag{3}$$

where $v_j = pf_j(t)$, and $\rho(x_1, x_2)$ is the distance between the points x_1, x_2 on the manifold, and a **strong** metric by the formula

$$r_M(v_1, v_2) = r_F(v_1, v_2) + \left|I(f_1(t)) - I(f_2(t))\right|. \tag{4}$$

Here $I(f(t))$ is the Riemannian length of the curve $f(t)$. Contracting deformations in the resulting spaces $\widehat{\Omega}(M^n)$ and $\widehat{\Omega}_M(M^n)$ can be constructed either by the method of steepest descent or by the known method based on the local uniqueness of geodesics.

2. Cycles modulo 2 of the loop space. Since the spaces $\widehat{\Omega}(M^n)$ and $\widehat{\Omega}(S^n)$ are homeomorphic, in order to study the homology of $\widehat{\Omega}(M^n)$ it is enough to compute the homology of the loop space on the sphere. We take the point $m_0 = (1, 0, \dots, 0)$ on the sphere $S^n : \ x_0^2 + \dots + x_n^2 = 1$ as the base point. Every great circle of the sphere S^n passing through the point m_0 is completely determined by a tangent unit vector e at the point m_0. We denote it by $S(e)$. Let $S(e)$ be an arbitrary oriented circle. Divide it into $(2k-1)$ equal parts by the points $m_0, m_1, \dots, m_{2k-2}$. In what follows we shall call the circle $S(e)$ the **supporting circle of the loop**.

Connect successively the points $m_0, m_k, m_1, m_{k+1}, \dots, m_{2k-2}, m_{k-1}, m_0$ by arcs of circles $\sigma_1, \dots, \sigma_{2k-1}$, so that the arc σ_i connects those points m_p and m_q for which

$$p \equiv (i-1)k \pmod{(2k-1)}; \qquad q \equiv ik \pmod{(2k-1)}. \tag{5}$$

Introducing on the constructed curve the reduced parameter, we obtain an oriented loop. The arcs $\sigma_i(e)$ will be called the **links of the loop**. If σ_i coincides with the shortest geodesic connecting m_p with m_q, we denote it by σ_i^-. In the case when σ_i coincides with the longer arc of the circle $S(e)$ connecting m_p and m_q, we denote it by σ_i^+. Under the p-projection an oriented loop passes to a $(2k-1)$-link nonoriented loop with the same supporting circle.

Definition 1. The set of $(2k-1)$-link nonoriented loops whose supporting circles lie on the sphere $S^{j+1}: \ x_0^2 + \dots + x_{j+1}^2 = 1$ forms a cycle $[j, 2k-1]$ modulo 2 of the space $\widehat{\Omega}^1(S^n)$. We note that $\dim[j, 2k-1] = (2k-1)(n-1) + j$.

3. Homology groups modulo 2 of the space of nonoriented loops.

Theorem 1. The cycles $[j, 2k-1]$ $(j = 0, 1, \dots, n-1; \ k = 1, 2, \dots)$ form a complete basic system of cycles modulo 2 of the space of nonoriented loops $\widehat{\Omega}(S^n)$.

The Betti numbers modulo 2 of the space $\widehat{\Omega}(S^n)$ are equal to:

$$\pi_2^s[\widehat{\Omega}(S^n)] = \begin{cases} 0, & \text{for } (2k-2)(n-1) < s < (2k-1)(n-1), \\ 1, & \text{for } (2k-1)(n-1) \le s \le 2k(n-1). \end{cases} \tag{6}$$

Proof. Since on S^n geodesic loops coincide with multiply repeated great circles, the set of extremal points in the space $\widehat{\Omega}(S^n)$ consists of a series of manifolds $P_1^{n-1}, P_2^{n-1}, \dots, \dots, P_k^{n-1}, \dots$, homeomorphic to projective space, and the length of a loop $v \in P_k^{n-1}$ is equal to $I(v) = 2\pi k$. Let $A_l = \{I(v) \le 2\pi l\}$ and $\varepsilon_k \ll 1$. The groups $H_s(A_{k+\varepsilon_k})$ stabilize for large k, and therefore $H_s(\widehat{\Omega}(S^n)) =$

$$= \lim_{k\to\infty} H_s(A_{k+\varepsilon_k}).$$

Thus, for the proof it is enough to compute, by induction, the homology groups of the domains $A_{k+\varepsilon_k}$.

Consider the exact sequence of the pair:

$$\cdots \xrightarrow{\Delta_*} H_s(A_{k-1+\varepsilon_{k-1}}) \xrightarrow{i_*} H_s(A_{k+\varepsilon_k}) \xrightarrow{j_*} H_s(A_{k+\varepsilon_k}, A_{k-1+\varepsilon_{k-1}}) \xrightarrow{\Delta_*} \cdots \tag{7}$$

Using contracting deformations, one can show that $A_{k-1+\varepsilon_{k-1}}$ is homotopy equivalent to $A_{k-\varepsilon_k}$, and $A_{k+\varepsilon_k}$ is homotopy equivalent to

$$A_{k-\varepsilon_k} \cup S(P_k^{n-1}, S_k).$$

By a continuous deformation in $\hat{\Omega}(S^n)$, the spherical neighborhood $S(P_k^{n-1}, \delta_k)$ can be deformed into the skew product B_k with base P_k^{n-1} and fiber the direct product

$$U_1^{n-1} \times \dots \times U_{4k-1}^{n-1}$$

of balls of dimension $(n-1)$. Hence it follows that the groups

$$H_s(A_{k+\varepsilon_k}, A_{k-1+\varepsilon_{k-1}})$$

are isomorphic to the groups

$$H_s(A_{k-\varepsilon_k} \cup B_k, A_{k-\varepsilon_k}).$$

Computing the index of the extremal manifold P_k^{n-1} in the open manifold B_k and applying the dual homology sequences of the triple (8), we obtain that the cycles

$$[j, 2k-1] \quad (j = 0, 1, \dots, n-1)$$

form a complete basis system of the groups

$$H_s(A_{k+\varepsilon_k}, A_{k-1+\varepsilon_{k-1}}).$$

Since $[j, 2k-1]$ are absolute cycles, the boundary homomorphism Δ_* in sequence (7) is trivial, whence it follows that the inclusion homomorphism

$$H_s(A_{k-1+\varepsilon_{k-1}}) \xrightarrow{i_*} H_s(A_{k+\varepsilon_k}) \tag{8}$$

is an isomorphism. The theorem is proved.

Let us note that for

$$k > \frac{s}{2(n-1)} + \frac{1}{2}$$

the s-dimensional homology groups of the domains $A_{k+\varepsilon_k}$ stabilize.

By the duality theorem there exists a basis system of cocycles

$$\{j, 2k-1\} \quad (j = 0, 1, \dots, n-1;\ k = 1, 2, \dots),$$

for which the scalar product

$$([i, 2l-1], \{j, 2k-1\}) = \delta_j^i \delta_k^l.$$

We turn to the computation of the cohomology ring $H(\hat{\Omega}(S^n))$. The natural embedding of the manifolds

$$[n-1, 2q-1] \to \hat{\Omega}(S^n)$$

induces inclusion homomorphisms of the cohomology rings

$$H([n-1, 2q-1]) \xleftarrow{p_*} H(\hat{\Omega}(S^n)); \tag{9}$$

therefore the computation of the ring $H(\hat{\Omega}(S^n))$ reduces to the study of the inclusion homomorphisms of homology groups

$$H_s([n-1, 2q-1]) \xrightarrow{i_*} H_s(\hat{\Omega}(S^n)) \tag{10}$$

and to the computation of the rings $H([n-1, 2q-1])$.

4. Homology of the manifolds $[n-1, 2q-1]$**.** Let the cycle

$$\langle j, a_1, \dots, a_{2q-1}\rangle \quad (0 \le j \le n-1;\; a_i = 1, +0, -0)$$

consist of p-parameter families of oriented loops for which: 1) when $j > 0$ the supporting circles lie on the sphere S^{j+1}, and when $j = 0$ the oriented circle

$$x_0^2 + x_1^2 = 1$$

serves as the supporting circle; 2) when $a_i = 1$, the arc $\sigma_i(e)$ is an arbitrary arc of the circle joining the points m_p and m_q; 3) when $a_i = +0$, the arc $\sigma_i(e) = \sigma_i^+(e)$; 4) when $a_i = -0$, the arc $\sigma_i(e) = \sigma_i^-(e)$. The dimension of the cycle is

$$\dim\langle j, a_1, \dots, a_{2q-1}\rangle = (a_1 + \dots + a_{2q-1})(n-1) + j.$$

We shall call a cycle **symmetric** if

$$a_i = a_{2q-i} \quad \text{for } i = 1, 2, \dots, q.$$

We shall call a symmetric cycle **marked** under the condition that, in the sequence $a_1, \dots, a_q$, the signs of the zeros alternate and the sign of the first zero in order is negative. We preliminarily divide nonsymmetric cycles into classes. The cycles

$$\langle j, a_1, \dots, a_{2q-1}\rangle, \quad \langle k, b_1, \dots, b_{2q-1}\rangle$$

belong to the same class if $k = j$ and either $a_i = \pm b_i$ for $1 \le i \le 2q-1$, or

$$a_i = \pm b_{2q-i}$$

for the same i. Then from each class we choose the cycle, called **marked**, for which the sign of the first zero is negative and the signs of the zeros alternate.

By the method of ([7]) one proves

Theorem 2. The marked nonsymmetric cycles $\langle j, a_1, \dots, a_{2q-1}\rangle$ ($j = 0, n-1$; $\sum a_i > 0$) and the marked symmetric cycles $\langle j, a_1, \dots, a_{2q-1}\rangle$ ($j = 0, 1, \dots, n-1$) form a complete basic system of cycles modulo 2 of the manifold $[n-1, 2q-1]$.

5. The cohomology ring of the space of nonoriented loops.

Theorem 3. Under the homomorphism of the embedding (10) the basic cycles are mapped as follows:

1) Nonsymmetric cycles

$$i_*\langle n-1, a_1, \dots, a_{2q-1}\rangle = 0;$$

$$i_*\langle 0, a_1, \dots, a_{2q-1}\rangle = 0, \qquad \text{if the sum } \sum a_i \text{ is even;}$$

$$i_*\langle 0, a_1, \dots, a_{2q-1}\rangle = [0, \sum a_i], \qquad \text{if the sum } \sum a_i \text{ is odd.}$$

2) Symmetric cycles ($0 \le j \le n-1$; $\sum a_i > 0$)

$$i_*\langle j, a_1, \dots, a_{2q-1}\rangle = 0, \qquad \text{if } \sum a_i \text{ is even;}$$

$$i_*\langle j, a_1, \dots, a_{2q-1}\rangle = [j, \sum a_i], \qquad \text{if } \sum a_i \text{ is odd.}$$

3)

$$i_*\langle n-1, -0, +0, \dots, +0, -0\rangle = 0.$$

Having computed the cohomology rings $H([n-1, 2q-1])$ and studied the associated homomorphisms of cohomology groups, we obtain the theorem:

Theorem 4. In the cohomology ring $H(\hat{\Omega}(S^n))$;

$$\{0, 2l-1\} * \{0, 2k-1\} = \left[\frac{(k+l-2)!}{(k-1)!(l-1)!}\right] \{n-1, 2(k+l)-3\}$$

$$(\text{for } k = 1, 2, \dots;\ l = 1, 2, \dots),$$

$$\{j, 2k-1\} * \{i, 2l-1\} = 0 \quad \text{in all other cases.}$$

Analogously to the way in which in § 17 of ([6]) the basic cocycles were constructed, in the space $\hat{\Omega}(S^n)/O$ one can construct a one-dimensional cocycle $\{1\}$, dual to the cycle $\langle 1, -0, +0, \dots, +0, -0\rangle$.

Theorem 5. The product of the cocycles $\{j, 2k-1\}$ of the basic system and the cocycle $\{1\}$ is determined by the multiplication table:

$$\{j, 2k-1\} * \{1\} = \{j+1, 2k-1\} \quad \text{for } j < n-1;$$

$$\{n-1, 2k-1\} * \{1\} = 0.$$

6. Estimates of the number of geodesic loops on a Riemannian manifold.

From Theorems 1 and 6, with the aid of ([6]), it follows:

Theorem 6. Let M^n be a Riemannian manifold of class C^3, homeomorphic to the n-dimensional sphere. Then: 1) on M^n there exists a countable sequence of series of geodesic loops with base point at $m_0 \in M^n$; 2) each series consists of n geodesic loops; 3) if the lengths C_p and C_q of two geodesic loops from one series coincide, then there exists a $(q-p)$-dimensional set of geodesic loops of equal length $C = C_p = C_q$.

Theorem 7. If M^n satisfies, in addition, the metric restriction of Morse ([5], [6])

$$0 < m \leq dl/ds \leq M < 2m,$$

then the lengths of the geodesic loops of the first series satisfy the inequalities

$$2\pi m \leq C_j \leq 2\pi M,$$

and among them there is no pair in which one geodesic is a multiple traversal of the other.

The example of the n-dimensional ellipsoid shows that the estimate found cannot be improved.

Remark. By the same method one can investigate the homology of a number of other functional spaces.

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Received
30 X 1963

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Note: Figure translations are in progress. See original paper for figures.