CYBERNETICS AND CONTROL THEORY

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1964

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Abstract

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SYNTHESIS OF LINEAR OPTIMAL SYSTEMS*

(Presented by Academician L. S. Pontryagin on 25 XI 1963)

A number of works have been devoted to the synthesis of linear optimal systems; among them we note the work of Neustadt (^{1}) and the work of N. N. Krasovskii (^{2}). In (^{1}) the synthesis problem for homogeneous linear systems is solved completely; in (^{2}) the general case of nonhomogeneous linear systems is considered, but the method proposed there is too complicated.

Here we describe a new synthesis method, suitable for arbitrary nonhomogeneous (nondegenerate, see below) linear systems. In connection with the method presented here, see also the work of Antosievich (^{3}).

1°. Formulation of the problem. Let the equation be given

$$\dot{x} = A(t)x + B(t)u + f(t). \tag{1}$$

Here x is an n-dimensional phase column vector; $A(t)$ is a summable $n \times n$-matrix (i.e., a matrix whose elements are summable on any bounded interval of the time axis); u is an r-dimensional control column vector; $B(t)$ is a summable $n \times r$-matrix; $f(t)$ is an n-dimensional summable column vector. The control u is sought in the class of measurable functions with values in a given convex compact polyhedron U of r-dimensional space, containing the origin.

The problem is as follows: for a given initial position x_0 in the phase space, find an **optimal control** that transfers the phase point along the corresponding (**optimal**) trajectory of equation (1) from x_0 to the origin in minimal time.

Write the equation

$$\dot{\psi} = -\psi A(t), \tag{2}$$

where ψ is an n-dimensional row, and define the "norm" $\|v\|$ in the space of r-dimensional row vectors v by the formula

$$\|v\| = \max_{u \in U} vu.$$

The necessary condition for optimality (the maximum principle, see ([4])) can now be formulated as follows.

For every optimal control $u(t)$, $0 \leq t \leq T$, there exists a nonzero solution $\psi(t)$, $0 \leq t \leq T$, of equation (2) such that almost everywhere on $0 \leq t \leq T$

$$\psi(t)B(t)u(t) = \|\psi(t)B(t)\|. \tag{3}$$

Equation (1) is assumed to be **nondegenerate** (see ([4])); this is equivalent to the assertion that, for any given nonzero solution $\psi(t)$ of equation (2), the control $u(t)$ is uniquely determined almost for all t from the maximum condition (3).

Thus, if from a given x_0 one can reach the origin, then the optimal problem (for this given x_0) will be solved if we can compute the initial value $\psi_{x_0} = \psi(0)$ of the corresponding solution $\psi(t)$ of equation (2). The computation of the vector ψ_{x_0} from the vector x_0 will be called the **synthesis** of the optimal system described by equation (1).

2°. **Derivation of the basic equation** (6) (see also (2, 3)). The solution $x(t)$ of equation (1) with the initial condition $x(0) = x_0$ has the form

$$x(t) = \Phi(t)\left[x_0 + \int_0^t \Phi^{-1}(\tau)(B(\tau)u(\tau) + f(\tau))\,d\tau\right],$$

where $\Phi(t)$ is the fundamental matrix for the homogeneous equation $\dot{x} = A(t)x$, normalized at $t = 0$. Let T_{x_0} be the optimal transition time from x_0 to the origin. Finding the optimal control $u_{x_0}(t)$, $0 \leq t \leq T_{x_0}$, is equivalent to solving the equation

$$z(T) = -\left(x_0 + \int_0^T \Phi^{-1}(t)f(t)\,dt\right) = \int_0^T \Phi^{-1}(t)B(t)u(t)\,dt = \int_0^T K(t)u(t)\,dt \tag{4}$$

with respect to the unknowns T, $u(t)$, $0 \leq t \leq T$, where T is taken to be the smallest positive root of this equation. The solution T_{x_0}, $u_{x_0}(t)$, $0 \leq t \leq T_{x_0}$, will be called the **optimal solution** of equation (4).

In order that, for any prescribed $T > 0$, equation (4) be solvable with respect to $u(t) \in U$, $0 \leq t \leq T$, it is necessary and sufficient that, for an arbitrary n-dimensional row χ, the inequality

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* The work was carried out in L. S. Pontryagin' s seminar on the theory of oscillations and automatic control.

$$\chi z(T) \leq \int_0^T \|\chi K(t)\| \, dt \tag{5}$$

hold.

The proof is given in (3).

Theorem. In order that equation (4) have an optimal solution T_{x_0}, $u_{x_0}(t)$, $0 \leq t \leq T_{x_0}$, it is necessary and sufficient that there exist a nonzero row ψ_0 satisfying the equation

$$\psi_0 z(T_{x_0}) = \int_0^{T_{x_0}} \|\psi_0 K(t)\| \, dt = \min_{\chi z(T_{x_0}) = \psi_0 z(T_{x_0})} \int_0^{T_{x_0}} \|\chi K(t)\| \, dt, \tag{6}$$

(i.e. the minimum is taken over all χ satisfying the condition $\chi z(T_{x_0}) = \psi_0 z(T_{x_0})$). Any solution ψ_0 of equation (6) may be taken as the vector ψ_{x_0} and the optimal control $u_{x_0}(t)$, $0 \leq t \leq T_{x_0}$, may be determined from the maximum condition (3), where $\psi(t)$, $0 \leq t \leq T_{x_0}$, is the solution of equation (2) with initial condition $\psi_{x_0} = \psi(0)$.

Proof. Let χ be an arbitrary n-dimensional row satisfying the condition $\chi z(T_{x_0}) > 0$, and let $\alpha \chi z(T_{x_0}) = \psi_0 z(T_{x_0})$; from the nondegeneracy of equation (1) it follows that the factor $\alpha > 0$. Therefore, from (6) there follows the inequality

$$\alpha \chi z(T_{x_0}) = \int_0^{T_{x_0}} \|\psi_0 K(t)\| \, dt \leq \int_0^{T_{x_0}} \|\alpha \chi K(t)\| \, dt,$$

i.e. inequality (5), equivalent to equation (4). If, conversely, T_{x_0}, $u_{x_0}(t)$, $0 \leq t \leq T_{x_0}$, is an optimal solution of equation (4), then, according to the maximum principle, there exists a solution $\psi(t) = \psi_{x_0} \Phi^{-1}(t)$ of equation (2) such that $\psi(t) B(t) u_{x_0}(t) = \|\psi_{x_0} K(t)\|$; consequently, multiplying (4) by ψ_{x_0}, we obtain

$$\psi_{x_0} z(T_{x_0}) = \int_0^{T_{x_0}} \psi_{x_0} K(t) u_{x_0}(t) \, dt = \int_0^{T_{x_0}} \|\psi_{x_0} K(t)\| \, dt,$$

i.e. equality (6).

It is easy to see that the control $u_0(t)$, $0 \leqslant t \leqslant T_{x_0}$, defined by the equation $\psi_0 K(t) u_0(t) = \|\psi_0 K(t)\|$, where ψ_0 is any nonzero solution of equation (6), is optimal: $u_0(t) = u_{x_0}(t)$, $0 \leqslant t \leqslant T_{x_0}$. Indeed, if $u_0(t) \neq u_{x_0}(t)$ on a set of positive measure, then

$$\psi_0 z(T_{x_0}) = \int_0^{T_{x_0}} \psi_0 K(t) u_{x_0}(t) \, dt < \int_0^{T_{x_0}} \psi_0 K(t) u_0(t) \, dt = \int_0^{T_{x_0}} \|\psi_0 K(t)\| \, dt,$$

which contradicts equality (6).

Thus, the synthesis problem is equivalent to solving equation (6) with respect to the unknowns ψ_0, T_{x_0}, and as T_{x_0} one must take the least positive root of this equation.

3°. **Solution of equation (6).** Equation (6) can be solved by the method of gradient descent, based on the following proposition.

For any $T > 0$, the gradient of the function $g(\chi) = \int_0^T \|\chi K(t)\|\,dt$ with respect to χ is continuous; every relative minimum of the function $g(\chi)$, under the condition $\chi z(T) = \mathrm{const} > 0$, is its absolute minimum (under the given condition $\chi z(T) = \mathrm{const}$).

Proof. In view of the nondegeneracy of equation (1),

$$\|\chi K(t)\| = \chi K(t) v_\chi(t),$$

where $v_\chi(t)$ is a piecewise constant function, depending on $\chi \neq 0$, on a set of full measure (in t), with values at the vertices of the polyhedron U; under a small change of χ, the function $v_\chi(t)$ changes on a set of small measure. Consequently,

$$\operatorname{grad} g(\chi) = \int_0^T K(t) v_\chi(t)\,dt$$

changes continuously together with χ. Let χ_1, χ_2 be two stationary points of the function $g(\chi)$ under the condition $\chi z(T) = \mathrm{const} > 0$; we shall show that $g(\chi_1) = g(\chi_2)$. Suppose the contrary, and let $g(\chi_1) > g(\chi_2)$. We have

$$\operatorname{grad} g(\chi_i) = \int_0^T K(t) v_{\chi_i}(t)\,dt = \lambda_i z(T),$$

$$\chi_i \cdot \operatorname{grad} g(\chi_i) = \int_0^T \chi_{iK}(t) v_{\chi_i}(t)\,dt = \int_0^T \|\chi_{iK}(t)\|\,dt = g(\chi_i) = \lambda_i \cdot \mathrm{const}, \quad i = 1, 2;$$

therefore $\lambda_1 > \lambda_2$.

Next we have:

$$\int_0^T K(t)(v_{\chi_1}(t) - v_{\chi_2}(t))\,dt = (\lambda_1 - \lambda_2) z(T);$$

multiplying both sides by χ_2, we obtain the relation

$$\int_0^T \chi_2 K(t) v_{\chi_1}(t)\, dt - \int_0^T \|\chi_2 K(t)\|\, dt = (\lambda_1 - \lambda_2) \cdot \mathrm{const} > 0,$$

which is contradictory, since

$$\int_0^T \chi_2 K(t) v_{\chi_1}(t)\, dt \leqslant \int_0^T \|\chi_2 K(t)\|\, dt.$$

The proposition just proved gives the following method for solving equation (6). Choose a "first approximation" χ_1 to the solution ψ_0, subjecting it to the sole condition $\chi_1 z(0) > 0$, and begin increasing the time t from 0 to the first instant t_1 (the "first approximation" to T_{x_0}) when

$$\chi_1 z(t_1) = \int_0^{t_1} \|\chi_1 K(t)\|\, dt$$

(if for every $t > 0$

$$\chi_1 z(t) > \int_0^t \|\chi_1 K(\tau)\|\, d\tau,$$

then the optimal problem with the given initial value x_0, obviously, has no solution). After this, by the method of gradient descent we find the minimum of the function $g_1(\chi) =$

$$= \int_0^{t_1} \|\chi K(t)\|\, dt$$

under the condition $\chi z(t_1) = \chi_1 z(t_1)$. If the minimum point $\chi_2 \neq \chi_1$, then

$$\chi_2 z(t_1) > \int_0^{t_1} \|\chi_2 K(t)\|\, dt,$$

and we shall begin to increase the time from the instant t_1 to the instant t_2, when again

$$\chi_2 z(t_2) = \int_0^{t_2} \|\chi_2 K(t)\|\, dt;$$

we obtain the "second approximations" t_2, χ_2, and so on. It is easy to see that the increasing sequence $t_1 \leq t_2 \leq \cdots$ has a finite least upper bound if and only if the optimal problem with the given initial value x_0 has a solution, and this least upper bound is equal to the optimal time T_{x_0}. In the case of finite T_{x_0}, the sequence of unit vectors

$$\frac{\chi_1}{\|\chi_1\|}, \quad \frac{\chi_2}{\|\chi_2\|}, \dots$$

($\|\chi\|$ is an arbitrary vector norm) converges to some compact set of vectors that constitute all unit-length solutions of equation (6).

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Received
24 X 1963

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Note: Figure translations are in progress. See original paper for figures.