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Abstract

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MATHEMATICS

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ROTATION OF THE FIELD IN ONE THEOREM OF THE THEORY OF STRUCTURALLY STABLE DYNAMICAL SYSTEMS ON THE TORUS

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As is well known, in many questions concerning dynamical systems in the plane

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

the use of "rotation of the field," i.e., consideration, along with system (1), of systems of the form

$$\dot{x} = P(x, y) - \mu Q(x, y), \qquad \dot{y} = Q(x, y) + \mu P(x, y), \tag{2}$$

proves to be very fruitful (). *In the theory of structurally stable systems in a plane domain, rotation of the field can be used in deriving a number of necessary conditions for structural stability (the absence, in a structurally stable system, of multiple limit cycles, separatrices going from a saddle to a saddle, etc.). In connection with such a use it is necessary to emphasize the following: structural stability of a dynamical system in a plane domain can naturally be defined in the following classes of right-hand sides: 1) in the class* C^n $(n = 1, 2, ...)$*; 2) in the class of analytic right-hand sides; 3) in the class of right-hand sides that are polynomials of a given degree* N .

Propositions of the theory of structurally stable systems that can be established by considering nearby systems of the form (2) are, obviously, **valid for all classes** 1), 2), 3). Rotation of the field can also be naturally used in the consideration of dynamical systems on surfaces, in particular in certain propositions of the theory of structurally stable dynamical systems. Structural stability of dynamical systems on surfaces, just as in the plane, should naturally be considered within classes of dynamical systems analogous to the classes 1), 2), 3) indicated for the plane **. However, in this case it is evidently necessary to introduce on surfaces functions analogous to polynomials. Whereas for orientable surfaces of genus $p \geq 2$ the question of such functions still remains open, in the case of an orientable surface of genus $p = 1$ (on the torus), upon introducing cyclic coordinates φ and ψ, it is natural to regard as such functions polynomials with respect to $\sin\varphi, \cos\varphi, \sin\psi, \cos\psi$, and the question of the roughness of dynamical systems on the torus can be posed within classes completely analogous to classes 1), 2), 3) in the plane. Therefore it is of interest to isolate those propositions of the theory of rough systems that can be established solely by using rotation of the field. New in comparison with propositions completely analogous to the theory of rough systems in the plane, in the theory of rough systems on surfaces, in particular on the torus, is the proposition on the absence, in rough systems, of nonclosed Poisson-stable semitrajectories. Below a proof of this proposition is given by using rotation of the field (i.e., a system of the form (2)).

Considering the torus developed onto the plane, we shall call congruent points those points whose coordinates coincide modulo 2π.

Let $x = x(t),\ y = y(t)$ be a Poisson-stable P^+ (P^-) semitrajectory, nonclosed on the torus. Then, as is known [3,4], the following propositions are valid:

Lemma 1. *The absolute values of the quantities $x(t), y(t)$, when the torus is developed onto the plane, increase without bound as $t \to +\infty$ $(-\infty)$.*

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* A priori necessary conditions of structural stability within class 1) are not necessary within classes 2) and 3). When the class of dynamical systems is narrowed, the derivation of necessary conditions for structural stability may become less trivial. The consideration of structural stability only within the class C^n, or even within the class C^∞, appears incomplete both from the mathematical point of view and from the point of view of applications (from the point of view of applications, structural stability within the class of analytic right-hand sides is essential). It should be noted that necessary conditions for structural stability in class 2) can be obtained by means of an elementary additional consideration using the Weierstrass theorem on approximation of functions of class C^n by polynomials. However, the definition of necessary conditions for structural stability in class 3) of right-hand sides becomes fundamentally more difficult, and in this class the necessary conditions for structural stability have not been established completely up to the present time: namely, the necessary conditions concerning **multiple limit cycles** have not been established. Conversely, it is obviously impossible to assert a priori the sufficiency of the structural-stability conditions derived in classes 2) and 3) for class 1). However, the proof of sufficiency of these conditions is the same in all classes.

** The study of structural stability of dynamical systems on surfaces in class 1) was carried out, for example, in ([2]). The question of structural stability in the other classes was not raised in ([2]).

Lemma 2. *If there are nonclosed Poisson-stable semitrajectories on the torus, then there can be no limit cycles on it that are not homologous to zero, and there exists at least one trajectory that is Poisson-stable in both directions.*

The proof of the following assertion presents no difficulty:

Theorem 1. *A dynamical system on the torus, rough in classes 1), 2), 3), has only a finite number of rough equilibrium states, a finite number of closed trajectories, and has no separatrices going from a saddle into a saddle.*

Theorem 2. *A dynamical system on the torus, rough in classes 1), 2), 3), cannot have nonclosed Poisson-stable semitrajectories.*

Proof. Suppose the contrary. Then, by Lemma 2, on the torus there exists a nonclosed trajectory L that is Poisson-stable in both directions,

$$x = \varphi(t), \qquad y = \psi(t). \tag{3}$$

Regarding the system of the form (1) as a system on the torus, consider the modified system (2). It is not hard to verify the validity of the following assertions:

I. If a dynamical system on the torus has only a finite number of rough equilibrium states and a finite number of limit cycles, then for any Poisson-stable trajectory one can specify so small an ε-neighborhood of it that, when the torus is developed onto the plane, this neighborhood contains no nodes, foci, or points of limit cycles of this system.

II. A trajectory of system (2), when the torus is developed onto the plane, may intersect a trajectory of system (1) at only one point, thereafter remaining above it for $\mu > 0$ or below it for $\mu < 0$. Without loss of generality one may assume that

$$\varphi(0) = \psi(0) = 0, \qquad P(0,0) > 0. \tag{4}$$

Obviously, there exists a sufficiently small real number $\rho > 0$ such that in the square

$$|x| \leqslant \rho/2, \qquad |y| \leqslant \rho/2 \tag{5}$$

the function $P(x,y) > 0$, and there is no equilibrium state and no point belonging to limit cycles of system (1). According to I, choose $\varepsilon > 0$ so small that in the strip

$$-\infty < x < +\infty, \qquad -\varepsilon/2 + \psi(t) \leqslant y \leqslant \psi(t) + \varepsilon/2 \tag{6}$$

there are no foci, nodes, or points of limit cycles of system (1).

Then, for any sufficiently small $\eta \in (0, \varepsilon/4)$, by virtue of the roughness of system (1), there is a number $\mu_0 > 0$ such that for each μ, $|\mu| < \mu_0$, there exists a trajectory L_μ of system (2), corresponding to the trajectory L of system (1), belonging to the strip

$$-\infty < x < +\infty, \qquad -\eta + \psi(t) \leq y \leq \psi(t) + \eta. \tag{7}$$

In this case the trajectory L_μ is an unclosed trajectory Poisson-stable in both directions. Fix an arbitrary value $\mu = \mu^* \in (0, \mu_0]$. By virtue of II, the following and only the following cases are possible: 1) the trajectory L_{μ^*} lies entirely above the trajectory L; 2) the trajectory L_{μ^*} intersects the trajectory L at only one point; 3) the trajectory L_{μ^*} lies entirely below the trajectory L. By transferring the origin of coordinates to a definite point of the trajectory L, or by replacing x and y by $-x$ and $-y$, we reduce all three cases to the single one: through the origin passes the semitrajectory L^+, corresponding on the torus to a Poisson-stable P^+ unclosed semitrajectory, in the η-neighborhood of which, above L^+, there passes the semitrajectory $L^+_{\mu^*}$, corresponding on the torus to a Poisson-stable P^+ unclosed semitrajectory, and in the square (5)

$$P(x, y) - \mu Q(x, y) > 0, \quad |\mu| < \mu_0.$$

Denote by A_0 and A_{μ^*} the points of intersection of the semitrajectories L^+ and $L^+_{\mu^*}$ with the line $x = -\rho/2$, and consider the set of trajectories $\{T\}$ of system (1) passing through the contact-free segment $A_0 A_{\mu^*}$. According to II, all trajectories $\{T\}$ lie entirely in the strip bounded by the semitrajectories L^+, $L^+_{\mu^*}$ and the straight-line segment $A_0 A_{\mu^*}$. Since in this strip there is no more than a countable set of saddles, there certainly exists a continuum of trajectories $\{T'\} \in \{T\}$ which are not their separatrices, i.e. such that $x(t) \to +\infty$ as $t \to +\infty$ (Lemma 1). Denote one of them by T^+. In this situation only two cases are possible:

A. In the family $\{T'\}$ there is no trajectory of system (1), other than L^+, that is Poisson-stable P^+ on the torus. Then, by virtue of the Poisson stability P^+ on the torus of the semitrajectory L^+, on the line $A_{\mu^*} A_0$, below the point A_0, and arbitrarily close to it, there exist points that are congruent to L^+.

B. In the family $\{T'\}$ there is at least one trajectory of system (1), distinct from L^+, that is Poisson-stable P^+ on the torus.* Then, in the strip bounded by the semitrajectories $L^+_{\mu^*}$, L^+ and the line $A_0 A_{\mu^*}$, there must be one more semitrajectory Poisson-stable P^+ on the torus. Consequently, there always exists a triple L^+_1, L^+, L^+_2 of trajectories Poisson-stable P^+ on the torus which, when the torus is unrolled onto the plane, have the property that L^+ lies between L^+_1 and L^+_2, and L^+_1, L^+_2 belong to the η-neighborhood of the semitrajectory L^+.

Suppose case A occurs. Between the semitrajectories L^+ and T^+ there are no saddles; otherwise, for a saddle $C(x_c, y_c)$ lying between them, its two ω-

separatrices and the contact-free segment A_1A_2, formed by the intersection of the line $x = \bar{x} > x_c$ with the semitrajectories T^+, L^+ (which exists because of the Poisson stability P^+ on the torus of the semitrajectory L^+), would form a “bag” containing the α-separatrix of the saddle $C(x_c, y_c)$, not emanating from any saddle of system (1) (Theorem 1). Consequently, in the strip (6) there are nodes, foci, or points of limit cycles, which contradicts the choice of the number ε. Denote by y_T^0 the ordinate of the point of intersection of the semitrajectory T^+ with the line $x = 0$, and, by virtue of the roughness of system (1), choose a number $\bar{\mu}_0 > 0$ so that, for any μ, $|\mu| < \bar{\mu}_0$, the corresponding points of trajectories of systems (1) and (2) are shifted by an amount not exceeding $y_T^0/2$. Denote by L_μ^+ and T_μ^+ the trajectories of system (2) corresponding to the semitrajectories L^+ and T^+ of system (1), for any fixed μ, $|\mu| < \mu_0$. They belong to the $y_T^0/2$-neighborhoods of the trajectories L^+ and T^+, respectively, and the point B_μ of intersection of the semitrajectory T_μ^+ with the line $x = 0$ has ordinate $y_\mu > 0$ for any μ, $|\mu| < \mu_0$, and the coordinate $x(t)$ of the semitrajectory T_μ^+ tends to $+\infty$ as $t \to +\infty$.

* These trajectories, distinct on the x, y plane, may on the torus be continuations of one another.

Between the semitrajectories L^+ and T_μ^+ there are still no saddles of system (1) or (2). Fix any value $\mu = \bar{\mu} \in (0, \mu_0)$ and, by the theorem on continuous dependence of solutions on initial conditions, for it choose $\delta > 0$ so small that all trajectories of system (2) passing through points of the segment $A_\delta : x = -\rho/2,\ y \in [-\delta + \psi(t^*), \psi(t^*)]$, where $\psi(t^*)$ is the ordinate of the point of intersection of the semitrajectory L^+ with the line $x = -\rho/2$, intersect the line $x = 0$ at values of the ordinates greater than zero. According to condition A, there exist two congruent points M^* and M^{**}, respectively on the segment A_δ and on the semitrajectory L^+. Consider, for $\mu = \bar{\mu}$, the two trajectories Γ^* and Γ^{**} of system (2) passing through them, which must necessarily be congruent. Since the point $O(0, 0)$ lies below the trajectory Γ^* and on the trajectory L^+, it follows that: 1) between Γ^{**} and L^+ there exists a point $\widetilde{M}(\tilde{x}, \tilde{y})$ congruent to the point $O(0, 0)$, and 2) the segment of the straight line $x = \tilde{x}$ between the semitrajectories L^+ and T_μ^+ is a segment without contact for the trajectories of systems (1) and (2). Introduce into consideration the trajectory θ_μ of system (2) passing through the point $O(0, 0)$ for any μ. For $\mu = 0$, θ_μ coincides with L^+, i.e., passes below the point $\widetilde{M}$, while for $\mu = \bar{\mu}$, θ_μ passes above the point $\widetilde{M}$, since it is above the trajectory Γ^{**}. Consequently, there exists a value $\mu^* \in (0, \bar{\mu})$ for which the trajectory θ_μ passes through the point $\widetilde{M}$ and, therefore, is a closed trajectory on the torus nonhomologous to zero, which is impossible by Lemma 2.

In considering case B, one can still show that there are no saddles between the semitrajectories L_1^+ and L_2^+. Denote by A_0, A_1, A_2 the points of intersection of the line $x = -\rho/2$ with the semitrajectories L^+, L_1^+, L_2^+, by y_1' and y_2' the ordinates of the points of intersection of the line $x = 0$ with the semitrajectories L_1^+ and L_2^+, and by $\bar{\eta}$ the minimum of the quantities $|y_1'|/2,\ |y_2'|/2$. By $\bar{\eta}$, in view of roughness, we find for system (2) an interval of variation of the parameter μ, $|\mu| < \mu_0$, such that the corresponding points of the trajectories of systems (1) and (2) would be displaced by an amount not exceeding $\bar{\eta}$. Denote by $L^+_{1\mu}$ and $L^+_{2\mu}$ the trajectories of system (2) corresponding to L^+_1 and L^+_2. Between them there are still no saddles, and moreover $L^+_{1\mu}$ passes above, and $L^+_{2\mu}$ below, the point $O(0,0)$ for any μ, $|\mu| < \mu_0$. Fix any $\mu = \bar{\mu} \in (0, \mu_0]$ and consider two values of μ: $\mu_1 = \bar{\mu}$, $\mu_2 = -\bar{\mu}$. By the theorem on continuous dependence of solutions on initial conditions, one can find a sufficiently small real number δ such that, for $\mu = \mu_1$, all trajectories of system (2) passing through points of the segment $A^+_\delta : x = -\rho/2,\ y \in [-\delta + \psi(t^*), \psi(t^*)]$, where $\psi(t^*)$ is the ordinate of the intersection of the semitrajectory L^+ with the line $x = -\rho/2$, intersect the axis $x = 0$ at ordinate values $y > 0$, while for $\mu = \mu_2$, all trajectories of system (2) passing through points of the segment $A^-_\delta : x = -\rho/2,\ y \in [\psi(t^*), \psi(t^*) + \delta]$ intersect the axis $x = 0$ at ordinate values $y < 0$. Consider the segment $A_\delta = A^+_\delta \cup A^-_\delta$. Because of Poisson stability on the torus, on the segment P^+ of the semitrajectory L^+ there must be found a point M^* to which there is a point M^{**} congruent on L^+. Setting in system (2) μ equal to $\bar{\mu}$ or $-\bar{\mu}$, respectively in the cases $M^* \in A^+_\delta$ or $M^* \in A^-_\delta$, the rest of the proof is carried out analogously to case A. The theorem is proved.

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