



Soviet-era science, translated into English

MATHEMATICS

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.18880>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

V. A. SOLONNIKOV

ON BOUNDARY-VALUE PROBLEMS FOR GENERAL LINEAR PARABOLIC SYSTEMS

(Presented by Academician V. I. Smirnov on 6 February 1964)

In the cylinder $Q = \Omega \times [0, T]$, where Ω is a bounded n -dimensional domain with smooth boundary S , consider the system of equations

$$\sum_{j=1}^m l_{kj} \left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j = f_k(x, t)$$

$$(k = 1, \dots, m; x = (x_1, \dots, x_n) \in \Omega, t \in [0, T]), \quad (1)$$

where l_{kj} are linear differential operators with complex coefficients that are smooth functions of x and t .

It is assumed that there exist integers s_k, t_k ($k = 1, \dots, m, \max_k s_k = 0$) and an integer $b > 0$ such that the following conditions are satisfied:

1. The degree of the polynomial $l_{kj}(x, t, i\xi\lambda, p\lambda^{2b})$ in the variable λ at each point $(x, t) \in Q$ does not exceed $s_k + t_j$; moreover, if $s_k + t_j < 0$, then $l_{kj} = 0$ (here $i = \sqrt{-1}$ and $\xi = (\xi_1, \dots, \xi_n)$).
2. Let l_{kj}^0 be the "principal part" of the polynomial l_{kj} , i.e., the sum of those terms of the polynomial l_{kj} which satisfy the condition

$$l_{kj}^0(x, t, i\xi\lambda, p\lambda^{2b}) = \lambda^{s_k + t_j} l_{kj}^0(x, t, i\xi, p),$$

and let $\mathcal{L}_0(x, t, i\xi, p)$ be the matrix with entries $l_{kj}^0(x, t, i\xi, p)$. The polynomial

$$L(x, t, i\xi, p) = \det \mathcal{L}_0(x, t, i\xi, p)$$

must have the following properties:

- a) $L(x, t, 0, p) = \gamma(x, t)p^2, \gamma \neq 0, r > 0$;
- b) the roots of the polynomial $L(x, t, i\xi, p)$ with respect to the variable p , at each point $(x, t) \in Q$ and for any real ξ , satisfy the inequality

$$\operatorname{Re} p_s < -\delta |\xi|^{2b} \quad (\delta > 0, s = 1, \dots, r).$$

By virtue of the homogeneity of the polynomials l_{kj}^0 ,

$$L(x, t, i\xi\lambda, p\lambda^{2b}) = \lambda^{\sum_{j=1}^m (s_j + t_j)} L(x, t, i\xi, p).$$

From property a) it follows that

$$\sum_{j=1}^m (s_j + t_j) = 2br.$$

Systems for which all these conditions are satisfied will be called **parabolic**. Systems parabolic in the sense of I. G. Petrovskii ⁽¹⁾ and in the sense of T. Shirota ⁽²⁾ are special cases of the general parabolic systems defined above. There exist general parabolic systems that do not belong to these classes.

We shall consider in the cylinder Q the problem consisting in finding a solution of system (1) satisfying the initial conditions

$$\sum_{j=1}^m C_{\alpha j} \left(x, \frac{\partial}{\partial t} \right) u_j \Big|_{t=0} = \varphi_\alpha(x) \quad (\alpha = 1, \dots, r), \quad (2)$$

and also the boundary conditions on the lateral surface $\Gamma = S \times [0, T]$ of the cylinder Q :

$$\sum_{j=1}^m B_{qj} \left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j \Big|_{\Gamma} = \Phi_q(x, t) \quad (q = 1, \dots, br). \quad (3)$$

Here $C_{\alpha j}$ and B_{qj} are linear differential operators with smooth coefficients, defined respectively in Ω and on Γ .

Suppose that there exist integers ρ_α ($\alpha = 1, \dots, r$) such that

$$C_{\alpha j}(x, p) = \begin{cases} C_{\alpha j}(x) p^{\frac{1}{2b}(\rho_\alpha + t_j)}, & \text{if } \frac{1}{2b}(\rho_\alpha + t_j) \text{ is a nonnegative integer,} \\ 0, & \text{in all other cases.} \end{cases}$$

The matrix $C(x, p)$ of size $r \times m$, with elements $C_{\alpha j}(x, p)$, must satisfy the following algebraic condition: the rows of the matrix

$$D(x, p) = C(x, p) \widehat{\mathcal{L}}_0(x, 0, 0, p),$$

where $\widehat{\mathcal{L}}_0$ is the adjugate matrix of \mathcal{L}_0 ($\widehat{\mathcal{L}}_0 = L\mathcal{L}_0^{-1}$), are linearly independent modulo the polynomial p^r .

The possibility of constructing a matrix C possessing the prescribed properties for any given parabolic system is determined by the topological structure of the domain Ω and by the matrix $\mathcal{L}_0(x, 0, 0, 1)$. If such a construction is possible, then the matrix C is determined uniquely up to certain algebraic transformations.

We now formulate the requirements imposed on the operators B_{qj} . Suppose that there exist integers σ_q such that the degree of

$$B_{qj}(x, t, i\xi\lambda, p\lambda^{2b})$$

with respect to λ does not exceed $\sigma_q + t_j$, and if $\sigma_q + t_j < 0$, then $B_{qj} = 0$. Let B_{qj}^0 be the sum of those terms of the polynomial $B_{qj}(x, t, i\xi, p)$ for which

$$B_{qj}^0(x, t, i\xi\lambda, p\lambda^{2b}) = \lambda^{\sigma_q + t_j} B_{qj}^0(x, t, i\xi, p).$$

Let $x \in S$; $\nu(x) = (\nu_1, \dots, \nu_n)$ be the unit vector of the inward normal to S ; and let $\zeta(x)$ be a vector lying in the tangent plane to S at the point x . It follows from the parabolicity condition that at each point $(x, t) \in \Gamma$, for any ζ and p satisfying the condition

$$\operatorname{Re} p > -\delta_1 |\zeta|^{2b}, \quad (4)$$

where $0 < \delta_1 < \delta$, and δ enters into the formulation of the parabolicity condition, the polynomial $L(x, t, i(\zeta + \tau\nu), p)$, as a function of τ , has br roots τ_s^+ with positive imaginary part and br with negative imaginary part. Let

$$M^+(x, t, \zeta, p, \tau) = \prod_{s=1}^{br} (\tau - \tau_s^+(x, t, \zeta, p)).$$

The matrix B_0 of size $br \times m$, with elements $B_{qj}^0(x, t, i\xi, p)$, must satisfy the following algebraic complementarity condition: at each point $(x, t) \in \Gamma$, for any tangent $\zeta(x)$ and any p satisfying condition (4), the rows of the matrix

$$A(x, t, i(\zeta + \tau\nu), p) = B_0(x, t, i(\zeta + \tau\nu), p) \widehat{\mathcal{L}}_0(x, t, i(\zeta + \tau\nu), p)$$

are linearly independent modulo the polynomial M^+ .

In addition, we impose one more algebraic requirement, which is probably connected with our approach to this problem. We assume that there exists a matrix $\widetilde{B}_0(x, t, i\xi, p)$ of size $br \times m$ possessing the following properties:

1) The elements $\widetilde{B}_{qj}^0(x, t, i\xi, p)$ are polynomials in ξ_j and p , and

$$\widetilde{B}_{qj}^0(x, t, i\xi\lambda, p\lambda^{2b}) = \lambda^{\mu_q + s_j} \widetilde{B}_{qj}^0(x, t, i\xi, p),$$

where the μ_q are integers.

2) The matrix \widetilde{B}_0 satisfies the complementing condition with the transposed matrix \mathcal{L}_0^* . If $n = 1$, then we may omit this requirement.

Let $l > 0$ be a noninteger, $C^l(\Omega)$ the class of functions defined and $[l]$ times continuously differentiable in Ω , whose derivatives of order $[l]$ satisfy the Hölder condition with exponent $l - [l]$, and $C_{x,t}^{l, \frac{1}{2b}l}(Q)$ the class of functions defined in Q , whose derivatives of order $[l]$ with respect to the variables x satisfy the Hölder condition in these same variables with exponent $l - [l]$, and, moreover, whose derivatives with respect to t of order $[\frac{l}{2b}]$ satisfy the Hölder condition in t with exponent $\frac{l}{2b} - [\frac{l}{2b}]$. The class $C_{x,t}^{l, \frac{1}{2b}}(\Gamma)$ is defined analogously. Norms in these classes are introduced in the usual way.

Under the conditions listed above, the following theorem holds.

Theorem. *Let l be a noninteger satisfying the condition $l > \max(0, \sigma_1, \dots, \sigma_{br})$. If the coefficients of the operators l_{kj} belong to the classes $C_{x,t}^{l-s_k, \frac{1}{2b}(l-s_k)}(Q)$, the coefficients of the operators $C_{\alpha j}$ to the classes $C^{l-\rho_\alpha}(\Omega)$, those of the operators B_{qj} to the classes $C_{x,t}^{l-\sigma_q, \frac{1}{2b}(l-\sigma_q)}(\Gamma)$, the surface S is a Lyapunov surface of class $C^{l+t_{\max}}$, $f_j \in C_{x,t}^{l-s_j, \frac{1}{2b}(l-s_j)}(Q)$, $\varphi_\alpha \in C^{l-\rho_\alpha}(\Omega)$, $\Phi_q \in C_{x,t}^{l-\sigma_q, \frac{1}{2b}(l-\sigma_q)}(\Gamma)$, and, finally, the functions $f_j, \varphi_\alpha, \Phi_q$ satisfy certain compatibility conditions determined by the system, the initial and the boundary conditions, then problem (1)–(3) has a unique solution $u = (u_1, \dots, u_m)$, with $u_j \in C_{x,t}^{l+t_j, \frac{1}{2b}(l+t_j)}(Q)$, for which the inequality*

$$\begin{aligned} \sum_{j=1}^m \|u_j\|_{C_{x,t}^{l+t_j, \frac{1}{2b}(l+t_j)}(Q)} &\leq C \left\{ \sum_{j=1}^m \|f_j\|_{C_{x,t}^{l-s_j, \frac{1}{2b}(l-s_j)}(Q)} + \right. \\ &\left. + \sum_{\alpha=1}^r \|\varphi_\alpha\|_{C^{l-\rho_\alpha}(\Omega)} + \sum_{q=1}^{br} \|\Phi_q\|_{C_{x,t}^{l-\sigma_q, \frac{1}{2b}(l-\sigma_q)}(\Gamma)} \right\}, \end{aligned}$$

holds, where the constant C does not depend on $u_j, f_j, \varphi_\alpha, \Phi_q$.

In the case where the integers t_j, s_j, ρ_α are divisible by $2b$, one can prove the existence of a solution $u = (u_1, \dots, u_m)$, with

$$u_j \in W_{px,t}^{2bl+t_j, l+\frac{t_j}{2b}}(Q),$$

for which the inequality

$$\sum_{j=1}^m \|u_j\|_{W_{px,t}^{2bl+t_j, l+\frac{t_j}{2b}}(Q)} \leq C' \left\{ \sum_{j=1}^m \|f_j\|_{W_{px,t}^{2bl-s_j, l-\frac{s_j}{2b}}(Q)} + \right.$$

$$\left. + \sum_{\alpha=1}^r \|\varphi_{\alpha}\|_{B_p^{2bl-\rho_{\alpha}-\frac{2b}{p}}(\Omega)} + \sum_{q=1}^{br} \|\Phi_q\|_{B_{p_x,t}^{2bl-\sigma_q-\frac{1}{p}, \frac{1}{2b}(2bl-\sigma_q-\frac{1}{p})}(\Gamma)} \right\}$$

holds; here l is an integer,

$$2bl > \max(0, \sigma_1, \dots, \sigma_{br}) + \frac{1}{p}.$$

Definitions of the norms entering this inequality may be found, for example, in (3).

The study of general boundary-value problems for systems that are parabolic in the sense of I. G. Petrovskii is the subject of works ⁴⁻⁹.

Received
29 I 1964

REFERENCES

- ¹ I. G. Petrovskii, *Bull. Moscow State Univ.*, Section A, **1**, 7 (1938).
- ² T. Shiota, *Osaka Math. J.*, **9**, No. 1 (1957).
- ³ V. A. Solonnikov, *DAN*, **138**, No. 4 (1961).
- ⁴ T. Ya. Zagorskii, *Ukr. Math. J.*, **9**, No. 3 (1957).
- ⁵ T. Ya. Zagorskii, *Mixed problems for systems of differential equations with each derivative of parabolic type*, Lvov, 1961.
- ⁶ S. D. Eidelman, *DAN*, **142**, No. 4 (1962).
- ⁷ S. D. Eidelman, *DAN*, **149**, No. 4 (1963).
- ⁸ L. N. Slobodetskii, *DAN*, **120**, No. 3 (1958).
- ⁹ M. S. Agranovich, M. I. Vishik, *UMN*, **18**, issue 1 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.